Lecture 2: The Consumer's Problem

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Goals

Students should be able to:

1. understand the Utility Maximization Problem (UMP) without using calculus.
2. derive the main characteristic of the Walrasian demands.
3. solve the UMP using calculus.
4. use the duality approach to obtain Walrasian and Hicksian demands.

Reading

Mandatory reading

- (JR) Sections 1.3 - 1.4

Suggested reading:

- (V) Chapters 3 and 4.

The main questions

Analysis of an economic model posed as a constrained maximization problem typically involves addressing four question:

1. Does the problem have a solution? (Existence)
2. What is the solution? (computation)
3. How can we characterize the solution? (characterization)
4. How does the solution vary with the parameters? (comparative statics or sensitivity analysis)

The consumer's problem

- Next Step: Construct a formal description of the humble atomistic consumer.
- Formally, the consumer seeks

$$\mathbf{x}^* \in B \quad \text{such that} \quad \mathbf{x}^* \succcurlyeq \mathbf{x} \quad \forall \mathbf{x} \in B$$

where $B \subset \mathbb{R}^n_+$ is the feasible set.

Some initial Assumptions

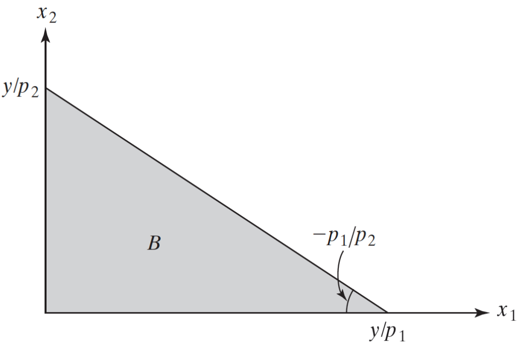
- Market economy:
  - $p_i > 0, i = 1, ..., n$
  - Individual consumer is an insignificant force on every market: the vector of market prices, $\mathbf{p} \gg \mathbf{0}$, as fixed from the consumer's point of view
- The consumer:
  - The consumer is endowed with a fixed money income $y \geq 0$.
  - We assume that $\mathbf{p} \cdot \mathbf{x} \leq y$
  - Budget set:

$$B(\mathbf{p}, y) = \left\{ \mathbf{x} | \mathbf{x} \in \mathbb{R}^n_+, \mathbf{p} \cdot \mathbf{x} \leq y \right\} \tag{1}$$

Some initial Assumptions

What does B look like?

Figure: Budget set, $B(\mathbf{p}, y) = \left\{ \mathbf{x} | \mathbf{x} \in \mathbb{R}^n_+, \mathbf{p} \cdot \mathbf{x} \leq y \right\}$, in the case of two commodities.



So, the consumer's problem is:

Formally, the consumer's utility-maximization problem (UMP) is written

$$\max_{\mathbf{x}\in\mathbb{R}^n_+} u(\mathbf{x}) \quad s.t \quad \mathbf{p}\cdot\mathbf{x} \leq y, \mathbf{x} \geq 0 \tag{2}$$

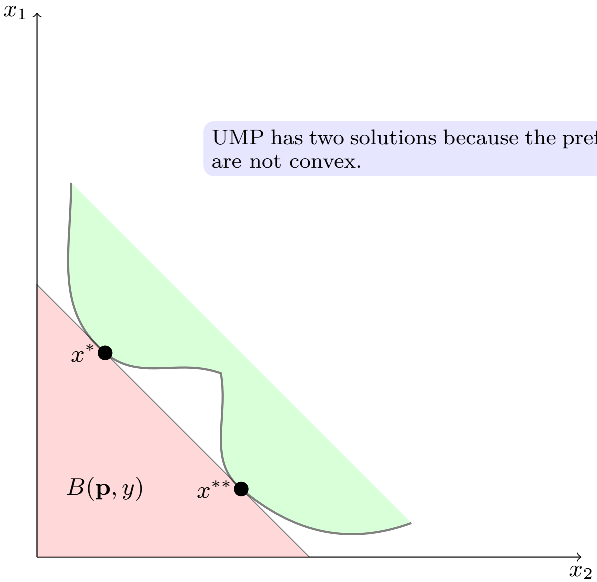
Note that if $\mathbf{x}^*$ solves this problem, then $u(\mathbf{x}^*) \geq u(\mathbf{x})$ for all $\mathbf{x} \in B(\mathbf{p}, y)$, which means that $\mathbf{x}^* \succcurlyeq \mathbf{x}$ for all $\mathbf{x} \in B(\mathbf{p}, y)$.

Potential Scenarios

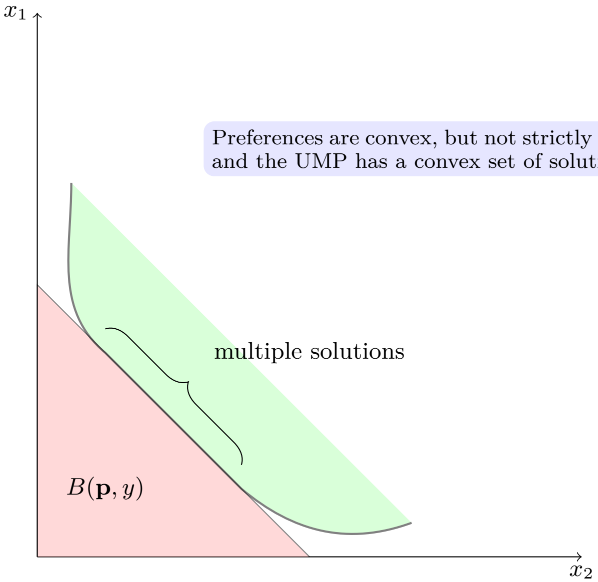
Remark

Recall that we need convex sets because convex sets are nice behave.

Potential Scenarios



Potential Scenarios



Basic Facts about the UMP

Propositions about UMP

Fix a continuous utility function $u(\cdot)$, and consider the UMP for various strictly positive prices $\mathbf{p}$ and nonnegative income levels y.

1. If $\mathbf{x}^*$ is a solution of the UMP for a given $\mathbf{p}$ and y, then $\mathbf{x}^*$ is also a solution for $(\lambda \mathbf{p}, \lambda y)$.
2. (Existence of Solution) The UMP for each $\mathbf{p}$ and y has at least one solution; some $\mathbf{x} \in B(\mathbf{p}, y)$ maximizes $u(\mathbf{x})$ over $B(\mathbf{p}, y)$.
3. (Unique Solution): If u is quasi-concave (preferences are convex), the set of solutions to the UMP for any $\mathbf{p}$ and y is convex. If u is strictly quasi-concave (if the preferences are strictly convex), then the UMP for each $\mathbf{p}$ and y has a unique solution.
4. If preferences are locally insatiable and if $\mathbf{x}$ is a solution to the UMP at $(\mathbf{p}, y)$, then $\mathbf{p} \cdot \mathbf{x} = y$.

Proof (1).

1. For any strictly positive λ, $\mathbf{p} \cdot \mathbf{x} \leq y \iff \lambda \mathbf{p} \cdot \mathbf{x} \leq \lambda y$;
2. Hence $B(\mathbf{p}, y) = B(\lambda \mathbf{p}, \lambda y)$.
3. The objective function, $u(\mathbf{x})$, doesn't change with λ;
4. Hence the set of solutions for $(\mathbf{p}, y)$ is the same as the set of solutions for $(\lambda \mathbf{p}, \lambda y)$.

□

Weierstrass Theorem

Theorem (Weierstrass Theorem: Sufficient conditions)

Let $f : A \to \mathbb{R}$ be a continuous function whose domain is a compact subset A in $\mathbb{R}^n$. Then, there exists a point $\mathbf{x}_m$ and $\mathbf{x}_M$ such that $f(\mathbf{x}_m) \leq f(\mathbf{x}) \leq f(\mathbf{x}_M)$ for all $\mathbf{x} \in A$; that is $\mathbf{x}_m \in A$ is the global min of f in A and $\mathbf{x}_M$ is the global max of f in A.

Example

Let $A = \mathbb{R}$ and $f(x) = x^3$ for all $x \in (0, 1)$. Then f is continuous, but A is not compact (it is closed, but not bounded). Since $f(A) = \mathbb{R}$, f evidently attains neither a maximum nor a minimum on A.

Proof: Existence of Solution.

- Claim: $B(\mathbf{p}, y)$ is nonempty. Trivial $\mathbf{0} \in B(\mathbf{p}, y)$.
- Claim: $B(\mathbf{p}, y)$ is is bounded: For any $\mathbf{x} \in B(\mathbf{p}, y)$ and for all i, $0 \leq x_i \leq y/p_i$.
- Claim: $B(\mathbf{p}, y)$ is closed: It is closed since it is defined by weak inequalities.
- Since $u(\mathbf{x})$ is continuous and $B(\mathbf{p}, y)$ is compact, then $u(\mathbf{x})$ attains a maximum on $B(\mathbf{p}, y)$ and the UMP has a solution.

□

Quasi-convexity and Optimization

Theorem (Quasi-convexity and Optimization)

Suppose $f : A \to \mathbb{R}$ be strictly quasi-concave where $A \subset \mathbb{R}^n$ is convex. Then, any local maximum of f on A is also a global maximum of f on A. Moreover, the set $\arg\max\{f(\mathbf{x}) : \mathbf{x} \in A\}$ of maximizers of f on A is either empty or a singleton.

Proof: Unique Solution.

- Claim: $B(\mathbf{p}, y)$ is convex:
  - If $\mathbf{x}', \mathbf{x}'' \in B(\mathbf{p}, y)$, then $\mathbf{px}' \leq y, \mathbf{px}'' \leq y, x_i \geq 0$ and $y_i \geq 0$ for all i.
  - Thus, for all $\theta \in [0, 1]$, $\mathbf{p}\left[\theta\mathbf{x}' + (1-\theta)\mathbf{x}''\right] = \theta\mathbf{px}' + (1-\boldsymbol{\theta})\mathbf{px}'' \leq y$, and
  - $\theta x_i' + (1-\theta)x_i'' \geq 0$ for all i,
  - that is, $\theta\mathbf{x}' + (1-\theta)\mathbf{x}'' \in B(\mathbf{p}, y)$.
- Then, the solution follows from the Quasi-convexity and Optimization's Theorem.

□

Proof: The set of solutions is convex.

Claim: If $\succcurlyeq$ is convex (u is quasi-concave), then the set of solutions for a choice from $B(\mathbf{p}, y)$ is convex.

- Assume that both $\mathbf{x}'$ and $\mathbf{x}''$ maximize $\succcurlyeq$ given $B(\mathbf{p}, y)$. That is, $\mathbf{x}'$ and $\mathbf{x}''$ belong to the set that maximize $\succcurlyeq$ given $B(\mathbf{p}, y)$.
- WTS that $\theta\mathbf{x}' + (1-\theta)\mathbf{x}''$ is convex.
- Since $B(\mathbf{p}, y)$ is convex, then $\theta\mathbf{x}' + (1-\theta)\mathbf{x}'' \in B(\mathbf{p}, y)$ (By definition of convexity of BC)
- By the convexity of the preferences: $\theta\mathbf{x}' + (1-\theta)\mathbf{x}'' \succcurlyeq \mathbf{x}' \succcurlyeq \mathbf{z}$ for all $\mathbf{z} \in B(\mathbf{p}, y)$.
- Thus, $\theta\mathbf{x}' + (1-\theta)\mathbf{x}'' \in \left\{\mathbf{x} : \max_{\mathbf{x}\in B(\mathbf{p},y)} u(\mathbf{x})\right\}$ is also a solution to the consumer problem.

□

The Walrasian (Marshalian) Demand Correspondence/Function

- The rule that assigns the set of optimal consumption vectors in the UMP to each price-income situation $(\mathbf{p}, y) \gg \mathbf{0}$ is denoted by $x(\mathbf{p}, y) \in \mathbb{R}^n$ is known as the Walrasian or ordinary market demand correspondence.
- When $x(\mathbf{p}, y) \in \mathbb{R}^n$ is single-valued for all $(\mathbf{p}, y)$, we refer to it as the Walrasian demand function.

The Walrasian (Marshalian) Demand Correspondence/Function

This proposition is a restatement of *"Propositions about UMP"*

Theorem (Properties of Walrasian Demand)

Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succcurlyeq$ defined on the consumption set $X = \mathbb{R}^n_+$. Then the Walrasian demand correspondence $\mathbf{x}(\mathbf{p}, y)$ possesses the following properties:

1. *Homogeneity of degree zero in $(\mathbf{p}, y)$: $\mathbf{x}(\alpha\mathbf{p}, \alpha y) = x(\mathbf{p}, w)$ for any $\mathbf{p}, y$ and scalar $\alpha > 0$.*
2. *Walras' law: $\mathbf{p} \cdot \mathbf{x} = y$ for all $\mathbf{x} \in \mathbf{x}(\mathbf{p}, y)$*
3. *Convexity/uniqueness: If $\succcurlyeq$ is convex, so that $u(\cdot)$ is quasi-concave, then $\mathbf{x}(\mathbf{p}, y)$ is convex set. Moreover, if $\succcurlyeq$ is strictly convex, so that $u(\cdot)$ is strictly quasi-concave, then $\mathbf{x}(\mathbf{p}, y)$ consists of a single element.*

Proof by contradiction:

Assume that a statement is not true then to show that that assumption leads to a contradiction. If P, Then Q, we assume P and Not Q, and then show that this "Not Q" leads to nonsense. Then we were wrong, so the statement must be true.

Proof: Walras's law.

Proof by contradiction.

- Assume that not, that is, $\mathbf{p} \cdot \mathbf{x}^* < y$.
- Therefore, there is an $\epsilon > 0$ such that $\mathbf{p}(x_1 + \epsilon, ..., x_n + \epsilon) < y$.
- By monotonicity, $(x_1 + \epsilon, ..., x_n + \epsilon) > \mathbf{x}^*$.
- Thus contradicting the assumption that $\mathbf{x}^*$ is optimal in $B(\mathbf{p}, y)$.
- So it must be true that $\mathbf{p} \cdot \mathbf{x}^* = y$

□

The consumer's problem

If we strengthen the requirements on $u(\mathbf{x})$ to include differentiability, we can use calculus methods to further explore the demand behavior.

K-T Procedure

The Optimization Problem

The problem is in general to maximize a (objective) function $f : \mathbb{R}^n_+ \to \mathbb{R}$ subject to a set of constraints $g_j : \mathbb{R}^n_+ \to \mathbb{R}$ for $j = 1, ..., m$. Assume that f and g_j are differentiable. The constrained maximization problem (with nonegativity constraints) is

$$\max_{\mathbf{x}} f(x_1, ..., x_n) \quad \text{subject to} \quad \begin{aligned} & g_1(x_1, ..., x_n) \geq 0 \\ & g_2(x_1, ..., x_n) \geq 0 \\ & \vdots \\ & g_m(x_1, ..., x_n) \geq 0 \\ & x_1 \geq 0, ..., x_n \geq 0 \end{aligned}$$

K-T Procedure that allows corner solutions

- Step 1: Set the Lagrangean:

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) \equiv f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x}) \equiv f(\mathbf{x}) + \boldsymbol{\lambda} \cdot g(\mathbf{x})$$

- Step 2: FOC: Exists an $(\mathbf{x}^*, \boldsymbol{\lambda}^*) \geq 0$, with $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ such that

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_i} \leq 0, \quad \frac{\partial \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_i} \mathbf{x}_i^* = 0, \quad i = 1, ..., n \tag{3}$$

$$g_j(\mathbf{x}^*) \geq 0, \quad \lambda_j^* g_j(\mathbf{x}^*) = 0, \quad j = 1, ..., m. \tag{4}$$

where:

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_i} = \frac{\partial f(\mathbf{x}^*)}{\partial x_i} + \sum_{j=1}^{m} \lambda_j^* \frac{\partial g_j(\mathbf{x}^*)}{\partial x_i}$$

Remark: This formulation (FOC) allows for corner solutions because we allows the possibility that $x_i^* = 0$.

Example (Substitute Goods)

Consider the following problem:

$$\max \; u(x_1, x_2) = x_1 + x_2 \quad \text{s.t} \quad B(\mathbf{p}, y) = \{y - p_1x_1 - p_2x_2 \geq 0, x_1 \geq 0, x_2 \geq 0\}$$

- Step 1: We set the lagrangean:

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = x_1 + x_2 + \lambda_1x_1 + \lambda_2x_2 + \lambda_3(y - p_1x_1 - p_2x_2) \tag{5}$$

  Note that we have the following restrictions:

$$\begin{aligned} g_1(x_1, x_2) &= x_1 \geq 0 \\ g_2(x_1, x_2) &= x_2 \geq 0 \\ g_3(x_1, x_2) &= y - p_1x_1 - p_2x_2 \geq 0 \end{aligned}$$

- Step 2: Does the problem has a solution?
- Step 3: Solve the system of inequalities. Which constraints are effective (bind)?
  - In principle there are a total of eight different combinations to be checked: $\emptyset, g_1, g_2, g_3, (g_1, g_2), (g_1, g_3), (g_2, g_3)$ and (g_1, g_2, g_3)
  - We can rule out the last one, because $g_1 = g_2 = 0 \implies g_3 > 0$. Since $u(\mathbf{x})$ is strictly increasing the individual spend all the money $\implies g_3 = 0$

Example (Substitute Goods)

Therefore we have only tree possible cases of effective constraints at the optimum: (g_1, g_3), (g_2, g_3) and g_3.

- Step 4: State FOC:

$$\frac{\partial \mathcal{L}^*}{\partial x_1} = 1 + \lambda_1 - p_1\lambda_3 \leq 0, \quad (1 + \lambda_1 - p_1\lambda_3)x_1 = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}^*}{\partial x_2} = 1 + \lambda_2 - p_2\lambda_3 \leq 0, \quad (1 + \lambda_2 - p_2\lambda_3)x_2 = 0 \tag{7}$$

$$x_1 \geq 0, \quad \lambda_1 x_1 = 0 \tag{8}$$

$$x_2 \geq 0, \quad \lambda_2 x_2 = 0 \tag{9}$$

$$(y - p_1x_1 - p_2x_2) \geq 0, \quad \lambda_3(y - p_1x_1 - p_2x_2) = 0 \tag{10}$$

- Step 5: Solve.

Example (Substitute Goods)

1. Case 1: g_3 is effective:
   - $g_3 = 0 \implies y - p_1 x_1 - p_2 x_2 = 0 \implies \lambda_3^* \geq 0$,
   - Moreover, g_3 the only effective implies that $x_1^* > 0 \implies \lambda_1^* = 0$ and $x_2^* > 0 \implies \lambda_2^* = 0$.
   - Because $x_1^* > 0$, from (6) $1 - p_1 \lambda_3 = 0 \implies \lambda_3^* = 1/p_1 > 0$.
   - Because $x_2^* > 0$, from (7) $1 - p_2 \lambda_3 = 0 \implies \lambda_3^* = 1/p_2 > 0$,
   - So, in this case we must have that $1/p_1 = 1/p_2$, so that $p_1 = p_2 = p$
   - Finally $x_1^* = (y - p x_2^*)/p_1$ and $x_2^* \in (0, y/p)$,
   - $(x_1^*, x_2^*, \lambda_1^*, \lambda_2^*, \lambda_3^*) = ((y - p x_2^*)/x_1, x_2^* \in (0, y/p), 0, 0, 1/p)$
2. Case 2: g_2, g_3 are effective:
   - $g_2 = 0 \implies x_2^* = 0 \implies \lambda_2 \geq 0$,
   - $g_3 = 0 \implies y - p_1 x_1 - p_2 x_2 = 0 \implies \lambda_3^* \geq 0$ and $x_1^* = y/p_1 \implies \lambda_1^* = 0$.
   - From (6): $1 + \lambda_2 - p_2 \lambda_3 = 0 \implies \lambda_3^* = 1/p_1$,
   - From (7): $\lambda_2^* \leq \frac{p_2}{p_1}$. So for $\lambda_2^* > 0$, we need $p_2 > p_1$.
   - Then: $(x_1^*, x_2^*, \lambda_1^*, \lambda_2^*, \lambda_3^*) = (y/p_1, 0, 0, p_2/p_1 - 1, 1/p_1)$
3. Case 3: g_1, g_3 are effective:
   - Similar to case 2: $(x_1^*, x_2^*, \lambda_1^*, \lambda_2^*, \lambda_3^*) = (0, y/p_2, 0, p_1/p_2 - 1, 0, 1/p_2)$

Example (Substitute Goods)

So, the Walrasian demand correspondence is:

$$\mathbf{x}^m(\mathbf{p}, y) = \begin{cases} (x_1^*, x_2^*) = (0, y/p_2) \quad \text{if} \quad p_1 > p_2 \\ (x_1^*, x_2^*) = (y/p_1, 0) \quad \text{if} \quad p_2 > p_1 \\ \{(x_1^*, x_2^*) : x_1^* + x_2^* = y/p, x_1 \geq 0, x_2 \geq 0\} \quad \text{if} \quad p_1 = p_2 \end{cases}$$

K-T Procedure for interior solution

If we are certain that $x_i > 0, \forall i = 1, 2, ..., n$, then there exists an $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$, with $\boldsymbol{\lambda}^* \in \mathbb{R}^m$, $\mathbf{x}^* > 0$ such that solve the following FOC:

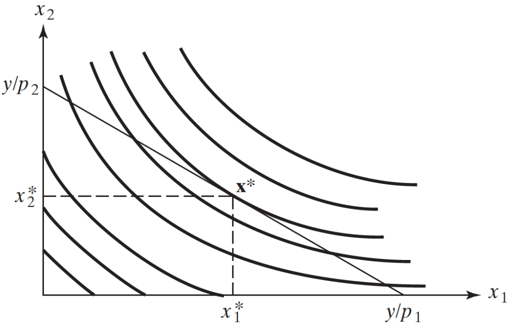
$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_i} = 0, \quad i = 1, ..., n \tag{11}$$

$$g_j(\mathbf{x}^*) \geq 0, \quad \lambda_j^* g_j(\mathbf{x}^*) = 0, \lambda_j^* \quad j = 1, ..., m. \tag{12}$$

You might be wondering....

How can we sure in microeconomics that the solution will be interior?

Figure: Interior Solution



Importance of Monotonicity

- If preferences are monotone then indifference curves are thin and downward sloping.
- Monotonicity also ensures the agent spend his entire budget (Walras's Law).
  - Commodities under study are goods in that increasing consumption increases utility: $\partial u/\partial x_i > 0$
  - At the maximizer, the multiplier λ cannot be zero; otherwise $\partial u/\partial x_i < 0$—a contradiction to our monotonicity assumption.
  - Since $\lambda > 0$ and $\lambda(y - \mathbf{p} \cdot \mathbf{x}) = 0 \implies \mathbf{p} \cdot \mathbf{x} = 0$.
  - So, the fact that marginal utilities are strictly positive implies that K-T conditions pick out the optimal solution.
  - If the non-negative constraints hold, the corresponding multipliers equal 0. Then, at a solution $\mathbf{x}^*$, we get:

$$\frac{\frac{\partial u(\mathbf{x}^*)}{\partial x_j}}{\frac{\partial u(\mathbf{x}^*)}{\partial x_k}} = \frac{p_j}{p_k}$$

A good question is: Under what circumstances can we ascertain that FOC is necessary for a local maxima, and that FOC is sufficient for LM?

Theorem (KT Conditions and Global Maxima)

Let f and $g_j, j = 1, ..., m$, all be concave functions defined on $\mathbb{R}^n$. Then FOC implies a global maximum condition.

Theorem (Arrow-Enthoven Theorem)

Let f and $g_j, j = 1, ..., m$, all be quasi-concave functions. Then conditions (FOC) is sufficient condition for a global maxima if any one of the following conditions are satisfied:

1. $\frac{\partial \mathbf{x}^*}{\partial x_i} < 0$, *for at least one variable x_i.*
2. $\frac{\partial \mathbf{x}^*}{\partial x_i} > 0$ *for some "relevant variable" i, where x_i is said to be a **relevant variable** if there exists an $\bar{x}$ in $S \equiv \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, g_j(\mathbf{x}) \geq 0, j = 1, ..., m\}$ such that $\bar{x}_i > 0$.*
3. $f'(\mathbf{x}) \neq 0$ *and $f(\mathbf{x})$ is twice continuously differentiable in a neighborhood of $\mathbf{x}^*$.*
4. *The function $f(\mathbf{x})$ is a concave function.*

- Functions g_j are all quasiconcave can be relaxed to the assumption that the constraint set $S = \equiv \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, g_j(\mathbf{x}) \geq 0, j = 1, ..., m\}$ is convex.
- "relevant variable" in UMP: there exists at least one commodity with which the consumer is not satiated at least in a neighborhood of $\mathbf{x}^*$.

Remark

So, if the utility function is quasi-concave and the feasible set is convex, then we do not need to check second order conditions (Bordered Hessian or Hessian)

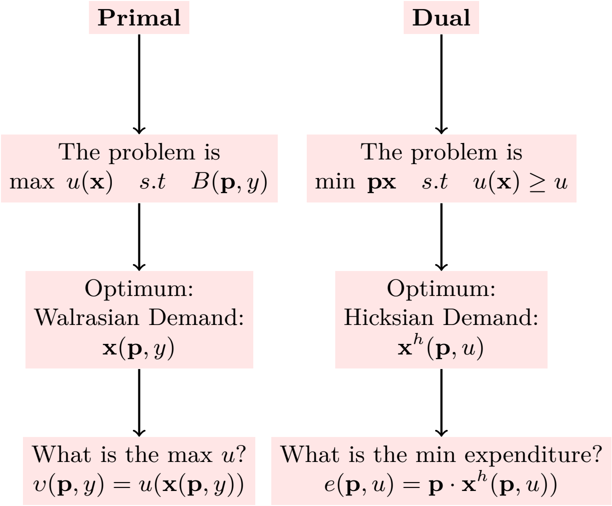
The consumer's problem

Theorem (Differentiable Demand)

Let $\mathbf{x}^* \gg \mathbf{0}$ *solve the consumer's maximization problem at prices* $\mathbf{p}^0 \gg \mathbf{0}$ *and income* $y^0 > 0$. *If*

- u *is twice continuously differentiable on* $\mathbb{R}^n_{++}$,
- $\partial u(\mathbf{x}^*)/\partial x_i > 0$ *for some* $i = 1, ..., n$, *and*
- *the bordered Hessian of* u *has non-zero determinant at* $\mathbf{x}^*$

then $\mathbf{x}(\mathbf{p}, y)$ *is differentiable at* $(\mathbf{p}^0, y^0)$


Primal
Dual
The problem is
max $u(\mathbf{x})$ $s.t$ $B(\mathbf{p}, y)$
The problem is
min $\mathbf{px}$ $s.t$ $u(\mathbf{x}) \geq u$
Optimum:
Walrasian Demand:
$\mathbf{x}(\mathbf{p}, y)$
Optimum:
Hicksian Demand:
$\mathbf{x}^h(\mathbf{p}, u)$
What is the max u?
$v(\mathbf{p}, y) = u(\mathbf{x}(\mathbf{p}, y))$
What is the min expenditure?
$e(\mathbf{p}, u) = \mathbf{p} \cdot \mathbf{x}^h(\mathbf{p}, u))$

Advice

Read Chapter 13 (Further Topics in Optimization) of Alpha Chiang. (Easy and intuitive)

The IUF

The relationship among prices, income, and the maximized value of utility can be summarized by a real-valued function $v : \mathbb{R}^n_+ \times \mathbb{R}_{++} \to \mathbb{R}$ defined as follows:

$$v(\mathbf{p}, y) = \max_{\mathbf{x} \in \mathbb{R}^n_+} \quad s.t \quad \mathbf{p} \cdot \mathbf{x} \leq y$$

The function $v(\mathbf{p}, y)$ is called the indirect utility function. It is the maximum-value function corresponding to the consumer's utility maximization problem.

Another definition of the IUF

Definition (IUF)

Given a continuous utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$, the indirect utility function $v : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by:

$$v(\mathbf{p}, y) = u(\mathbf{x}^*(\mathbf{p}, y)) \quad \text{where} \quad \mathbf{x}^*(\mathbf{p}, y) \in \arg\max_{\mathbf{x} \in B(\mathbf{p}, y)} u(\mathbf{x}).$$

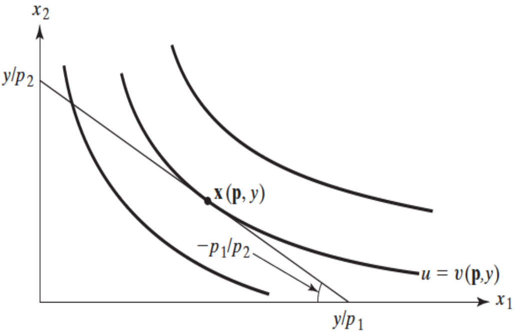
This indirect utility function measures changes in the optimized value of the objective function as the parameters (prices and wages) change and the consumer adjusts her optima consumption accordingly.

The IUF

- When $u(\mathbf{x})$ is continuous, $v(\mathbf{p}, y)$ is well-defined for all $\mathbf{p} \gg \mathbf{0}$ and $y \geq 0$ because a solution to the UMP is guaranteed to exist
- If $u(\mathbf{x})$ is strictly quasiconcave, then the solution is unique and we write it as $\mathbf{x}(\mathbf{p}, y)$, the consumer's demand function.
- The maximum level of utility that can be achieved when facing prices $\mathbf{p}$ and income y therefore will be that which is realized when facing prices $\mathbf{p}$ and income y therefore will be that which is realized when $\mathbf{x}(\mathbf{p}, y)$ is chosen. Hence

$$v(\mathbf{p}, y) = u(\mathbf{x}(\mathbf{p}, y)).$$

Figure: Indirect utility at prices $\mathbf{p}$ and income y.



Properties of IUF

Theorem (Properties of the indirect Utility Function)

Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succcurlyeq$ defined on the consumption set $X = \mathbb{R}^n_+$. The indirect utility function $v(\mathbf{p}, y)$ is

1. *Homogeneous of degree zero in $(\mathbf{p}, y)$,*
2. *Strictly increasing in y,*
3. *nonincreasing in $\mathbf{p}$,*
4. *Quasi-convex in $(\mathbf{p}, y)$.*
5. *Continuous in $\mathbf{p}$ and w.*

If $v(\mathbf{p}, y)$ is differentiable at $(\mathbf{p}^0, y^0)$ and $\partial v(\mathbf{p}^0, y^0)/\partial y \neq 0$, then,

1. *Roy's identity:*

$$x_i(\mathbf{p}^0, y^0) = -\frac{\partial v(\mathbf{p}^0, y^0)/\partial p_i}{\partial v(\mathbf{p}^0, y^0)/\partial y}, i = 1, ..., n. \tag{13}$$

Proof: Homogenous of degree zero.

WTS that $v(\mathbf{p}, y) = v(t\mathbf{p}, ty)$ for all $t > 0$

- By definition of IUF

$$v(t\mathbf{p}, ty) = [\max \ u(\mathbf{x}) \quad \text{st} \quad t\mathbf{p} \cdot \mathbf{x} \leq t\mathbf{y}]$$

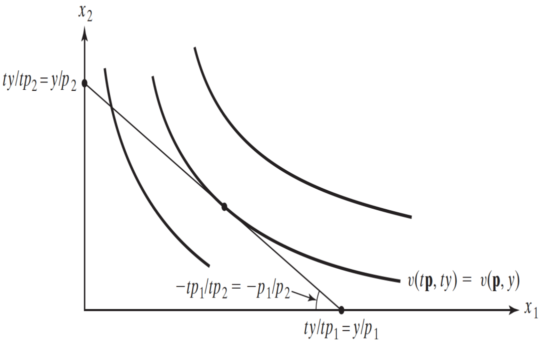
- But, this is equal to

$$v(t\mathbf{p}, ty) = [\max \ u(\mathbf{x}) \quad \text{st} \quad \mathbf{p} \cdot \mathbf{x} \leq \mathbf{y}]$$

- So, $v(\mathbf{p}, y) = v(t\mathbf{p}, ty)$.

□

Figure: Homogeneity of the indirect utility function in prices and income



Proof: v is nonincreasing in $\mathbf{p}$.

- Let $\mathbf{p}' > \mathbf{p} \gg \mathbf{0}$.
- Since $\mathbf{p}' > \mathbf{p}$ and strictly positive vectors, for any $\mathbf{x} > \mathbf{0}$, we have that $\mathbf{p}'\mathbf{x} \geq \mathbf{p}\mathbf{x}$ (In words: when $\mathbf{x}$ is affordable with prices $\mathbf{p}'$ is true also that $\mathbf{x}$ is affordable when the prices are lower)
- Thus, if $\mathbf{p}'\mathbf{x} \leq y \implies \mathbf{p}\mathbf{x} \leq y$.
- Hence, $B(\mathbf{p}', y) \subset B(\mathbf{p}, y)$.
- We conclude that

$$v(\mathbf{p}', y) = \left[\max\ u(\mathbf{x}) \quad \text{st} \quad \mathbf{p}' \cdot \mathbf{x} \leq y\right] \leq v(\mathbf{p}, y) = [\max\ u(\mathbf{x}) \quad \text{st} \quad \mathbf{p} \cdot \mathbf{x} \leq y]$$

- So $v(\mathbf{p}', y) \leq v(\mathbf{p}, y)$ whenever $\mathbf{p}' > \mathbf{p} \gg \mathbf{0}$

□

Proof: v is nondecreasing in y.

- Let $y' > y$.
- Since $y' > y$, then $B(\mathbf{p}, y) \subset B(\mathbf{p}, y')$.
- We conclude that

$$v(\mathbf{p}, y) = [\max \; u(\mathbf{x}) \quad \text{st} \quad \mathbf{p} \cdot \mathbf{x} \leq y] \leq v(\mathbf{p}, y') = \left[\max \; u(\mathbf{x}) \quad \text{st} \quad \mathbf{p} \cdot \mathbf{x} \leq y'\right]$$

- So $v(\mathbf{p}, y) \leq v(\mathbf{p}, y')$ whenever $y' > y$

□

Proof: Quasiconvex.

Recall: f is quasi-convex $\iff f[\alpha\mathbf{x}+(1-\alpha)\mathbf{y}] \leq \max[f(\mathbf{x}), f(\mathbf{y})]$. So, WTS that $v(\mathbf{p}^\alpha, y^\alpha) \leq \max[v(\mathbf{p}, y), v(\mathbf{p}', y')] \quad \forall\alpha \in [0,1]$.

- Pick any two points: $(\mathbf{p}, y)$ and $(\mathbf{p}', y')$
- Take the convex combination between these two points:

$$\mathbf{z} = (\mathbf{p}^\alpha, y^\alpha) = (\alpha\mathbf{p} + (1-\alpha)\mathbf{p}', \alpha y + (1-\alpha)y')$$

- Let $\mathbf{x}^\alpha = \mathbf{x}(\mathbf{p}^\alpha, y^\alpha)$. Then $\mathbf{p}^\alpha\mathbf{x}^\alpha \leq y^\alpha$. IOW: $\mathbf{x}^\alpha \in B^\alpha$
- Note that:

$$\begin{aligned} \mathbf{p}^\alpha \cdot \mathbf{x}^\alpha &\leq y^\alpha \\ \left[\alpha\mathbf{p} + (1-\alpha)\mathbf{p}'\right] \cdot \mathbf{x}^\alpha &\leq \alpha y + (1-\alpha)y' \\ 0 &\leq \alpha(y - \mathbf{p}\mathbf{x}^\alpha) + (1-\alpha)(y' - \mathbf{p}'\mathbf{x}^\alpha) \end{aligned}$$

- So, either $y - \mathbf{p}\mathbf{x}^\alpha \geq 0 \implies y \geq \mathbf{p}\mathbf{x}^\alpha$ or $y' \geq \mathbf{p}'\mathbf{x}^\alpha$, or possible both,
- This implies that at least one of the following two things are true:
  - $y \geq \mathbf{p}\mathbf{x}^\alpha \implies v(\mathbf{p}^\alpha, y^\alpha) \leq v(\mathbf{p}, y)$, or
  - $y' \geq \mathbf{p}'\mathbf{x}^\alpha \implies v(\mathbf{p}^\alpha, y^\alpha) \leq v(\mathbf{p}', y')$.
- So, it must be true that $v(\mathbf{p}^\alpha, y^\alpha) \leq \max[v(\mathbf{p}, y), v(\mathbf{p}', y')] \quad \forall\alpha \in [0,1]$

□

Value Functions

- We often encounter optimization problems like:

$$\max_{\mathbf{x}} f(\mathbf{x}, \boldsymbol{\theta}) \quad s.t. \quad g(\mathbf{x}, \boldsymbol{\theta}) = 0 \quad \text{and} \quad \mathbf{x} \geq \mathbf{0}.$$

  where
  - $\mathbf{x}$: vector of choice variables.
  - $\boldsymbol{\theta} = (\theta_1, ..., \theta_m)$ is a vector of parameters that may enter the objective function, the constraint, or both.
  - The solution to this problem will depend upon $\boldsymbol{\theta}$: $\mathbf{x}^*(\boldsymbol{\theta})$
- Let $\phi(\boldsymbol{\theta}) = f(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta})$: it gives the value achieved by the objective function. (maximum value function)

How does $\phi(\boldsymbol{\theta})$ changes as we change one or more parameters of the problem?

Theorem (The envelope theorem)

Consider

$$\max_{\mathbf{x}} f(\mathbf{x}, \boldsymbol{\theta}) \quad s.t. \quad g(\mathbf{x}, \boldsymbol{\theta}) = 0 \quad and \quad \mathbf{x} \geq \mathbf{0}.$$

and suppose that the objective function and constraint are continuously differentiable in $\boldsymbol{\theta}$. *For each* $\boldsymbol{\theta}$, *let* $\mathbf{x}^*(\boldsymbol{\theta}) \gg \mathbf{0}$ *uniquely solve the problem and assume that it is also continuously differentiable in in* $\boldsymbol{\theta}$. *Let* $\mathcal{L}(\mathbf{x}, \boldsymbol{\theta}, \lambda)$ *be the problem's associated Lagrangian function and let* $(\mathbf{x}(\boldsymbol{\theta}), \lambda(\boldsymbol{\theta}))$ *solve the KT conditions. Finally, let* $\phi(\boldsymbol{\theta})$ *be the problem's associated maximum-value function. Then, the Envelope Theorem states that:*

$$\frac{\partial \phi(\boldsymbol{\theta})}{\partial \theta_j} = \left. \frac{\partial \mathcal{L}}{\partial \theta_j} \right|_{\mathbf{x}(\boldsymbol{\theta}), \lambda(\boldsymbol{\theta})} \qquad j = 1, ..., m.$$

Value Functions

- So, we do not need to reformulate all the problem in order to know how $\phi(\boldsymbol{\theta})$ changes when we move θ_j.
- Example: How much vary $u(\mathbf{x}^*(\mathbf{p}, y))$ when we move y?

$$\frac{\partial v(\mathbf{p}, y)}{\partial y} = \left.\frac{\partial \mathcal{L}}{\partial y}\right|_{\mathbf{x}(\mathbf{p},y),\lambda(\mathbf{p},y)} = \lambda(\mathbf{p}, y)$$

- So, $\lambda(\mathbf{p}, y)$ is known as the marginal utility of income, or shadow price of income.

In general the jth Lagrangian multiplier signifies the marginal rate of change of the optimal value of the objective function with respect to a change in the jth constraint.

Proof: Roy's Identity using Calculus.

We will make use of the envelope theorem.

- Let $\mathbf{x}^*(\mathbf{p}, y)$ be the strictly positive solution to the K-T FOC,
- There must exists $\lambda^* > 0$.
- Applying the Envelope theorem:

$$\frac{\partial v(\mathbf{p}, y)}{\partial p_i} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial p_i} = -\lambda^* x_i^*.$$

- But $\lambda^* = \frac{\partial v(\mathbf{p}, y)}{\partial y} > 0$. Hence

$$x_i^* = x_i(\mathbf{p}, y) = -\frac{\partial v(\mathbf{p}, y)/\partial p_i}{\partial v(\mathbf{p}, y)/\partial y}, i = 1, ..., n.$$

□

Example

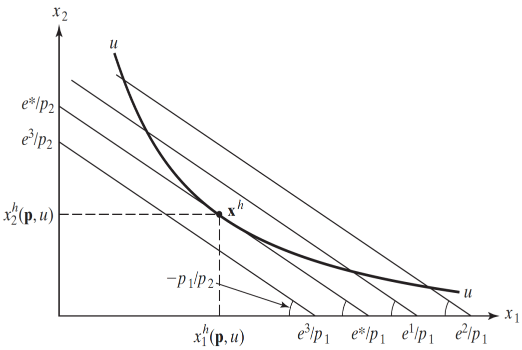
Using our previous example:

$$\boldsymbol{v}(p_1, p_2, y) = \max\{y/p_1, y/p_2\}$$

The expenditure function

- Powerful way to summarize a great deal about the consumer's market behavior.
- (IUF): We fixed market prices and income, and sought the maximum level of utility the consumer could achieve.
- Expenditure function: we fixed prices, what is the minimum level of money expenditure the consumer must take facing a given set of prices to achieve a given level of utility?

Figure: Finding the lowest level of expenditure to achieve utility level u.



The Expenditure Minimization Problem (EMP)

We define the expenditure function as the minimum-value function

$$e(\mathbf{p}, u) = \min_{\mathbf{x} \in \mathbb{R}^n_+} \mathbf{p} \cdot \mathbf{x} \quad s.t \quad u(\mathbf{x}) \geq u \tag{14}$$

for all $\mathbf{p} \gg 0$ and all attainable utility level u. For future reference, let $\mathcal{U} = \{u(\mathbf{x}) | \mathbf{x} \in \mathbb{R}^n_+\}$ denote the set of attainable utility levels. Thus, the domain of $e(\cdot)$ is $\mathbb{R}^n_{++} \times \mathcal{U}$

Properties of Expenditure Function

Proposition: Properties of Expenditure Function

Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succcurlyeq$ defined on the consumption set $X = \mathbb{R}^n_+$. The expenditure function $e(\mathbf{p}, u)$ is

1. Homogeneous of degree one in $\mathbf{p}$.
2. Strictly increasing in u and nondecreasing in $\mathbf{p}$.
3. Concave in $\mathbf{p}$.
4. Continuous in $\mathbf{p}$ and u.

If, in addition, $\succcurlyeq$ is strictly convex ($u(\cdot)$ is strictly quasi-concave), and $e(\mathbf{p}, u)$ is differentiable in $\mathbf{p}$ at $(\mathbf{p}^0, u^0)$ with $\mathbf{p}^0 \gg \mathbf{0}$, then

1. Shepard's lemma:

$$\frac{\partial e(\mathbf{p}^0, u^0)}{\partial p_i} = x_i^h(\mathbf{p}^0, u^0), \quad i = 1, ..., n.$$

Homework: Prove this Proposition.

Hicksian Demands

Properties of Hicksian Demand

Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succcurlyeq$ defined on the consumption set $X = \mathbb{R}^n_+$. Then for any $\mathbf{p} \gg \mathbf{0}$, the Hicksian demand correspondence $\mathbf{x}^h(\mathbf{p}, u)$ possesses the following properties:

1. Homogeneity of degree zero in $\mathbf{p}$: $\mathbf{x}^h(\alpha\mathbf{p}, u) = h(\mathbf{p}, y)$ for any $\mathbf{p}, u$ and $\alpha > 0$,
2. No excess utility: For any $\mathbf{x} \in \mathbf{x}^h(\mathbf{p}, u)$, $u(\mathbf{x}) = u$.
3. Convexity/uniqueness: If $\succcurlyeq$ is convex, then $\mathbf{x}^h(\mathbf{p}, u)$ is a convex set; and if $\succcurlyeq$ is strictly convex, so that $u(\cdot)$ is strictly quasi-concave, then there is a unique element in $\mathbf{x}^h(\mathbf{p}, u)$.

The Expenditure Minimization Problem (EMP)

The following proposition describes the formal relationship between EMP and the UMP following to MGW

Relationship between EMP and UMP (MGW)

Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^n_+$ and that the price vector $\mathbf{p} \gg \mathbf{0}$. We have

1. If $\mathbf{x}^*$ is optimal in the UMP when income $y > 0$, then $\mathbf{x}^*$ is optimal in the EMP when the required utility level is $u(\mathbf{x}^*)$. Moreover, the maximized expenditure level in this EMP is exactly y.
2. If $\mathbf{x}^*$ is optimal in the EMP when the required utility level is $u > u(\mathbf{0})$, then $\mathbf{x}^*$ is optimal in the UMP when income is $\mathbf{p} \cdot \mathbf{x}^*$. Moreover, the maximized utility level in this UMP is exactly u.

The Expenditure Minimization Problem (EMP)

The following proposition describes the formal relationship between EMP and the UMP following to JR

Relationship between EMP and UMP (JR)

Let $v(\mathbf{p}, y)$ and $e(\mathbf{p}, u)$ be the indirect utility function and expenditure function for some consumer whose utility function is continuous and strictly increasing. Then for all $\mathbf{p} \gg \mathbf{0}$, $y \geq 0$, and $u \in \mathcal{U}$

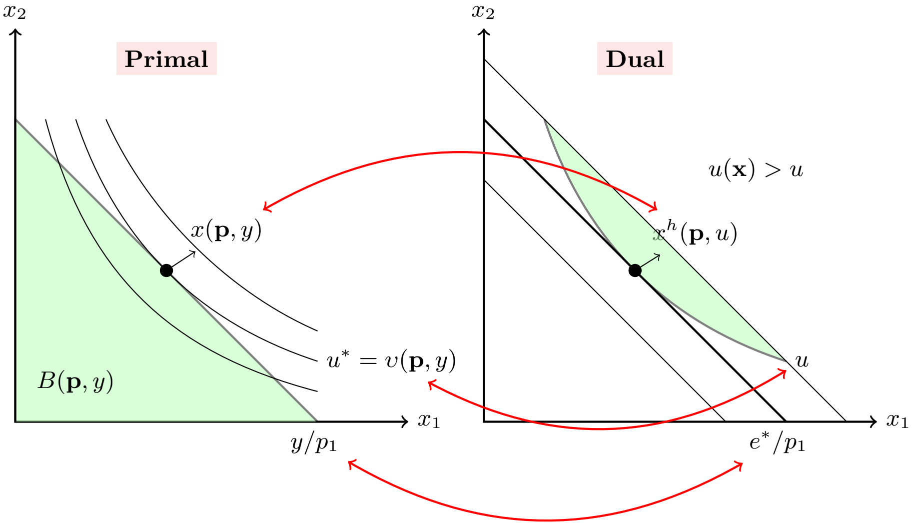
1. $e(\mathbf{p}, v(\mathbf{p}, y)) = y$
2. $v(\mathbf{p}, e(\mathbf{p}, y)) = u$

Duality Between Walrasian and Hicksian Demand Functions

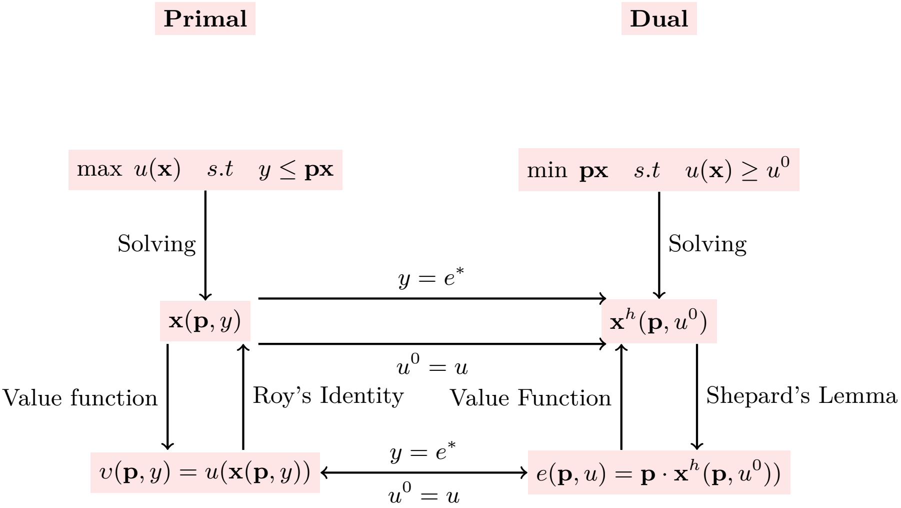
Duality Between Walrasian and Hicksian Demand Functions

Assume that $u(\cdot)$ is continuous, strictly increasing, and strictly quasi-concave on $\mathbb{R}^n_+$. Then for all $\mathbf{p} \gg \mathbf{0}$, $y \geq 0$, and $u \in \mathcal{U}$

1. $x_i(\mathbf{p}, y) = x_i^h(\mathbf{p}, v(\mathbf{p}, y))$
2. $x_i^h(\mathbf{p}, u) = x_i(\mathbf{p}, e(\mathbf{p}, u))$



x_2
Primal
$x(\mathbf{p}, y)$
$u^* = v(\mathbf{p}, y)$
$B(\mathbf{p}, y)$
x_1
y/p_1
x_2
Dual
$u(\mathbf{x}) > u$
$x^h(\mathbf{p}, u)$
u
x_1
e^*/p_1


Primal
Dual
max $u(\mathbf{x})$ $s.t$ $y \leq \mathbf{px}$
min $\mathbf{px}$ $s.t$ $u(\mathbf{x}) \geq u^0$
Solving
Solving
$y = e^*$
$\mathbf{x}(\mathbf{p}, y)$
$\mathbf{x}^h(\mathbf{p}, u^0)$
$u^0 = u$
Value function
Roy's Identity
Value Function
Shepard's Lemma
$y = e^*$
$v(\mathbf{p}, y) = u(\mathbf{x}(\mathbf{p}, y))$
$e(\mathbf{p}, u) = \mathbf{p} \cdot \mathbf{x}^h(\mathbf{p}, u^0))$
$u^0 = u$

Example on blackboard