Lecture 1:
Consumer Theory: Preferences and Utility

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Microeconomía Avanzada I

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Goals

1. To understand preferences relations over bundles, their mathematical representation, and different properties that preferences can satisfy.
2. To examine under which conditions and individual's preference relation can be mathematically represented with a utility function.
3. To understand the relationship between preferences and utility.

Reading

Mandatory reading

- (JR) Sections 1.1 and 1.2

Suggested reading:

- (JR) Appendix A1 and A2.
- (V) Chapters 3 and 4.

What is the idea?

- From your first class of microeconomic theory you probably know the concept of utility.
  - $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$,
  - $u(x_1, x_2) = x_1 + x_2$,
  - $u(x_1, x_2) = \min\{x_1, x_2\}$, etc.
- In principle, we could begin by assuming the existence of utility function and examining its properties.
- However, if we want to know what exactly is involved in such assumption, we can try to find a set of axioms of choice, the acceptance of which is equivalent to the existence of a utility function.
- In the following slides we will see (formally) what is behind the utility function. Let's start with some primitive notions.

Primitive Notions

Primitive Notions

Definition (Consumption set)

Consumption set (X) or choice set: it represents the set of all alternatives, or complete consumption plans, that the consumer can conceive, whether some of them will be achievable in practice or not.

Primitive Notions

Definition (Consumption bundle)

Let $x_i \in \mathbb{R}$ represent the number of units of good i. We assume that only nonnegative units of each good are meaningful and that it is always possible to conceive of having no units of any particular commodity. Further, we assume there is a finite, fixed, but arbitrary number n of different goods. We let $\mathbf{x} = (x_1, ..., x_n)$ be a vector containing different quantities of each of the n commodities and call $\mathbf{x}$ a **consumption bundle** or a **consumption plan**

- A consumption bundle $\mathbf{x} \in X$ is thus represented by a point $\mathbf{x} \in \mathbb{R}^n_+$
- Sometimes we think of the consumption set as the entire nonnegative orthant, $X = \mathbb{R}^n_+$
- Good properties of $\mathbb{R}^n_+$!

Convex sets

- Convex sets are the basic building blocks in virtually every area of microeconomic theory (and optimization)
- In theoretical work, convexity is most often assumed to guarantee that the analysis is mathematically tractable and that the results are clear-cut and "well-behaved".

Definition (Convex sets in $\mathbb{R}^n$)

$S \subset \mathbb{R}^n$ is a convex set if for all $\mathbf{x}^1 \in S$ and $\mathbf{x}^2 \in S$, we have

$$t\mathbf{x}^1 + (1-t)\mathbf{x}^2 \in S \tag{1}$$

for all t in the interval $0 \leq t \leq 1$.

Convex sets

- Consider $S \subset \mathbb{R}^2_+$

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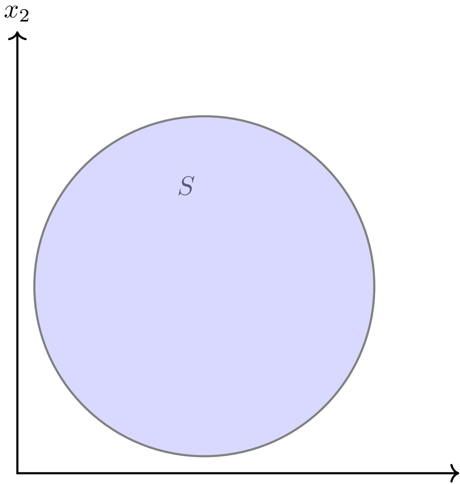

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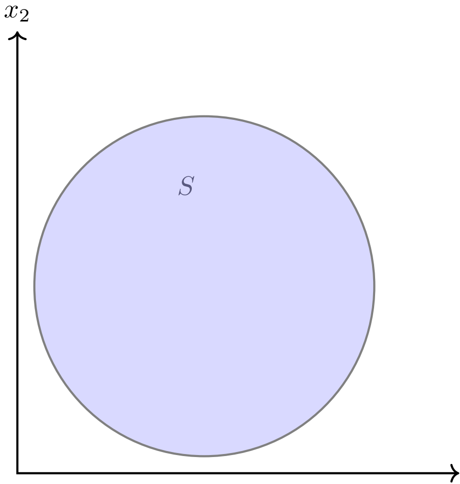

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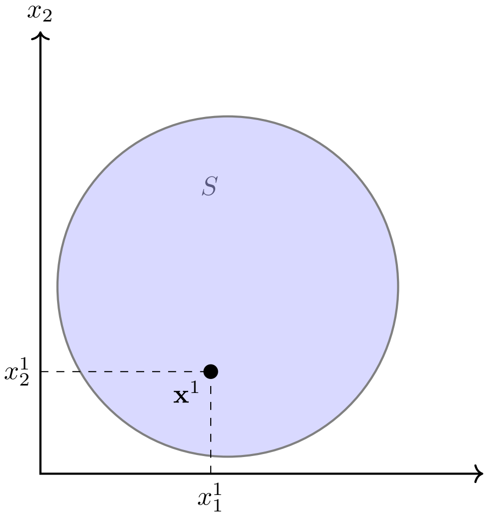

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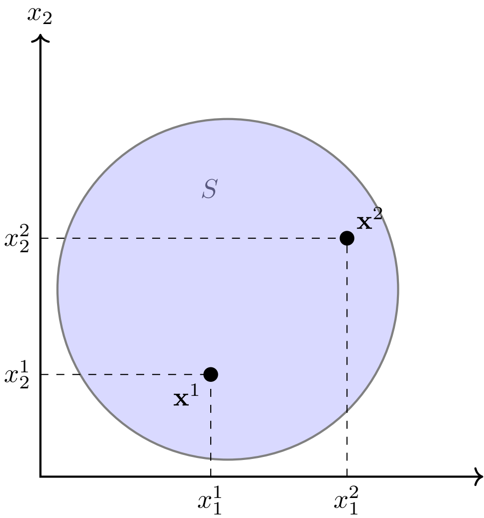

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- Pick any to points $\mathbf{x}^1 \in S$ and $\mathbf{x}^2 \in S$,
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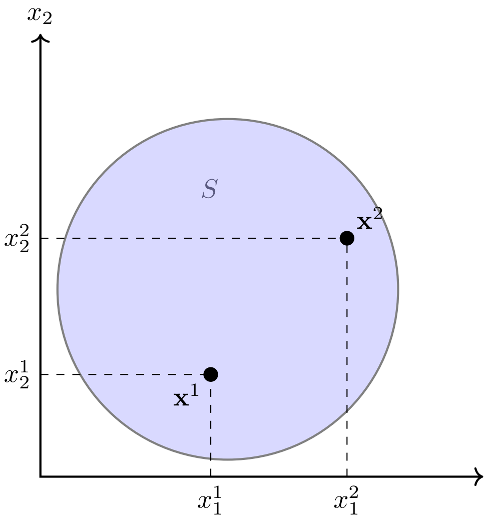

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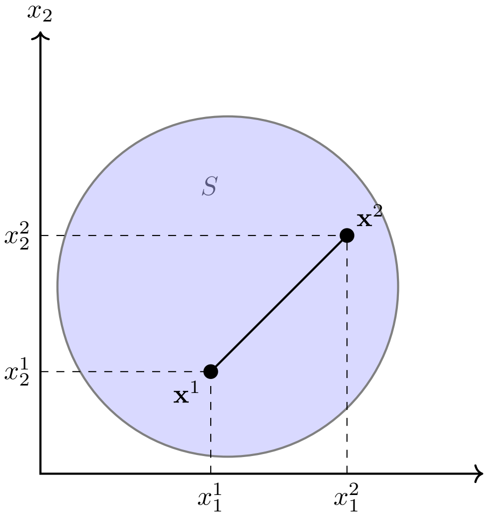

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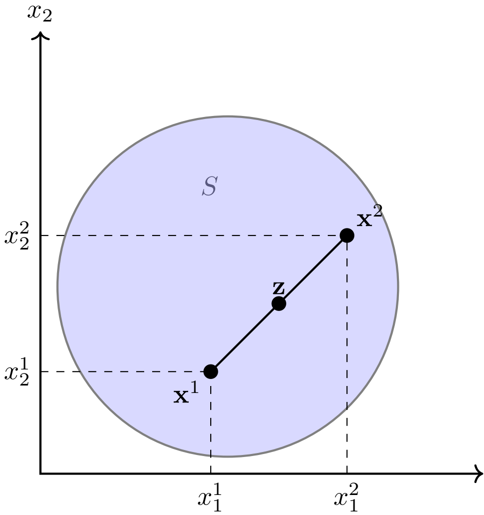

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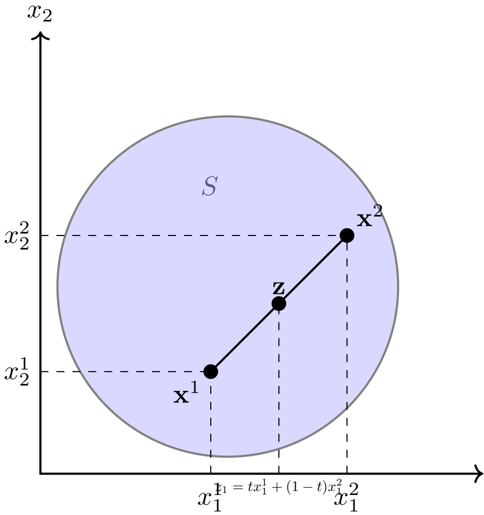

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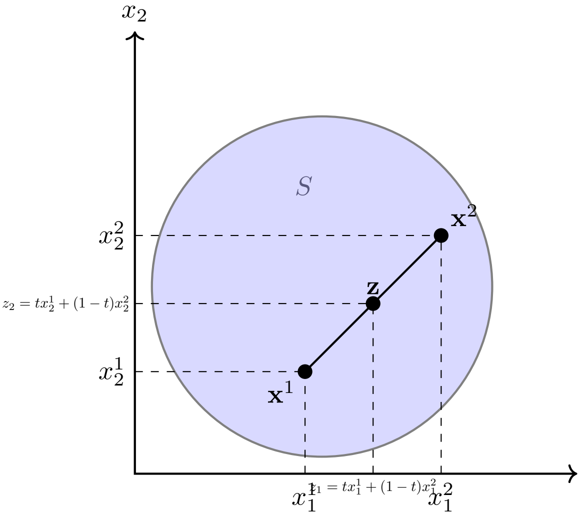

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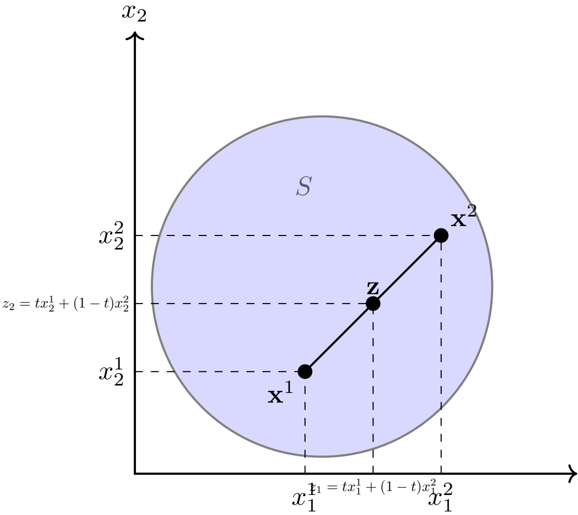
- Is $\mathbf{z}$ inside of S?



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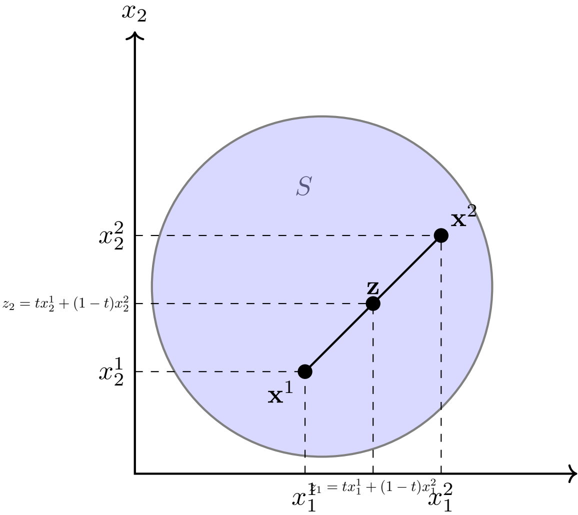
- Is $\mathbf{z}$ inside of S?
- Can we do the same for all $\mathbf{x}^1 \in S$ and $\mathbf{x}^2 \in S$?



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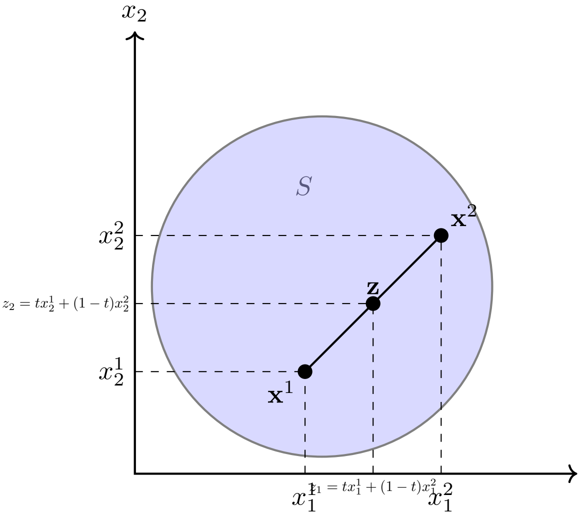
- Is $\mathbf{z}$ inside of S?
- Can we do the same for all $\mathbf{x}^1 \in S$ and $\mathbf{x}^2 \in S$?
- Then, we have a convex set.



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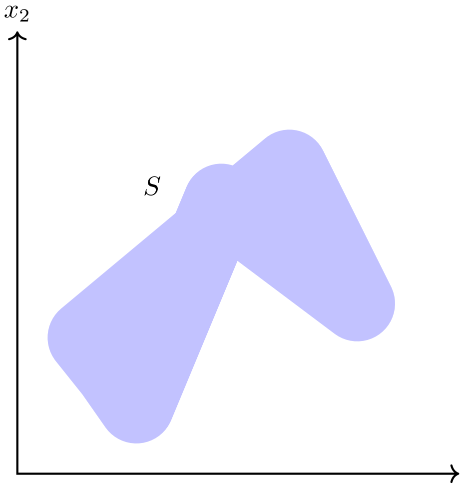

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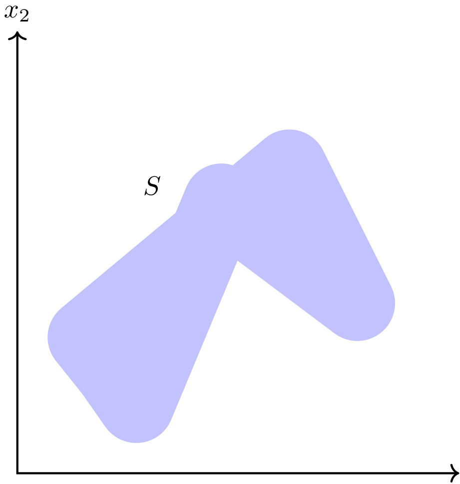

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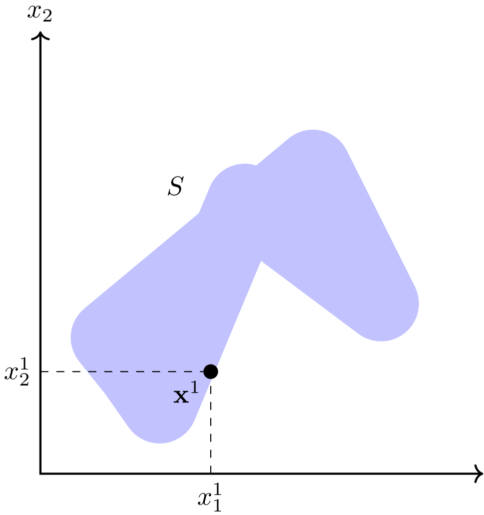

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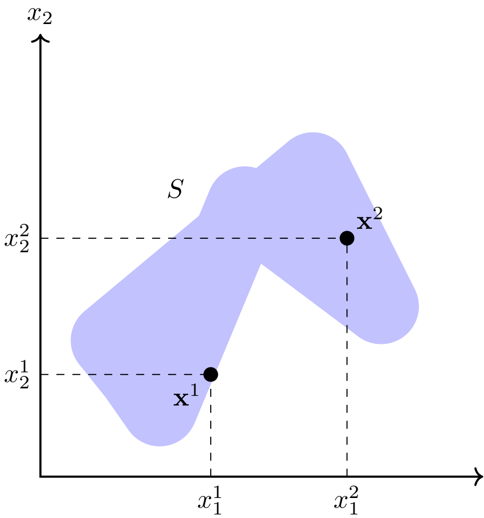

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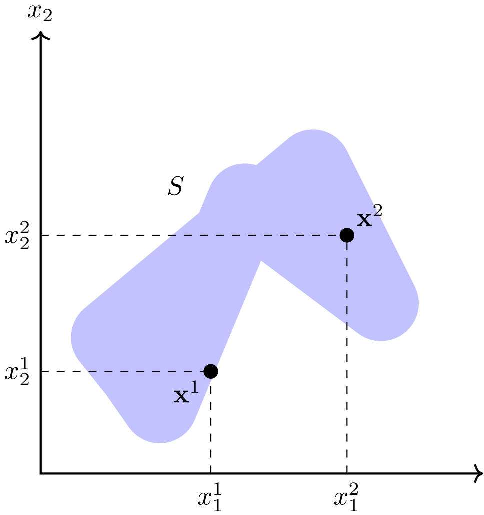

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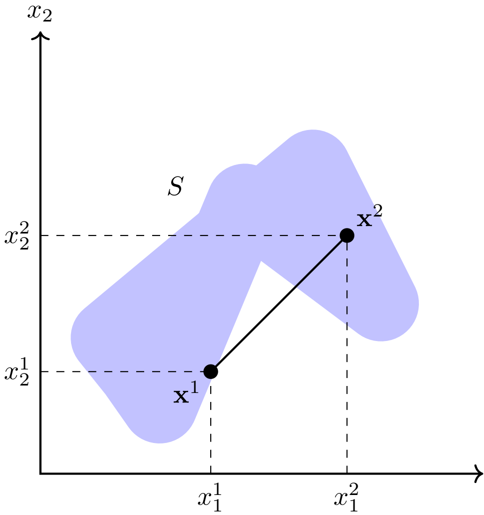

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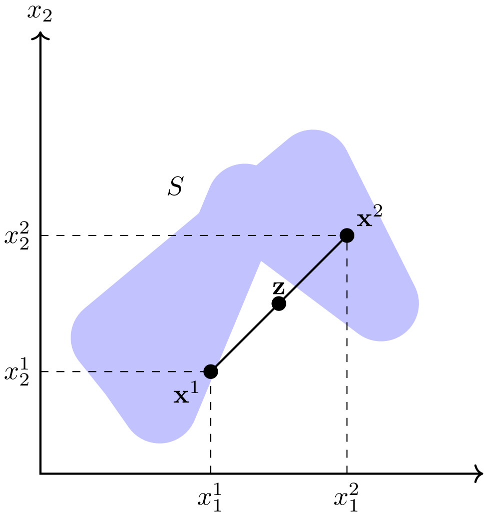

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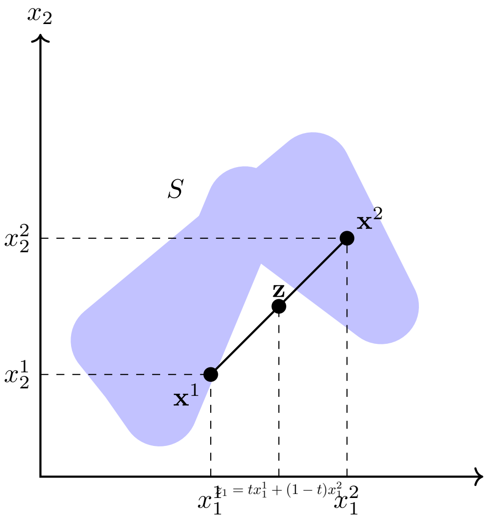

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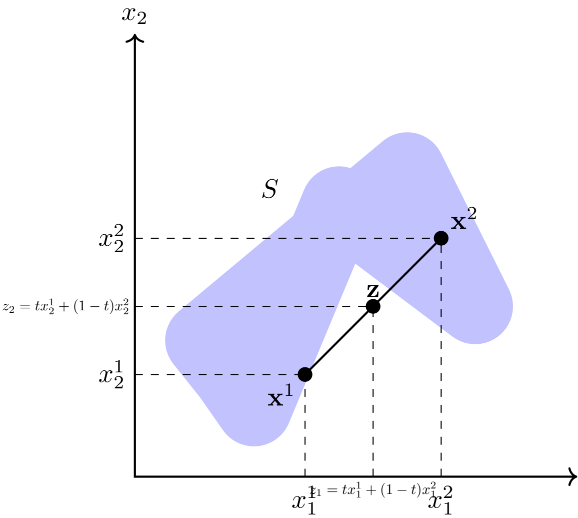

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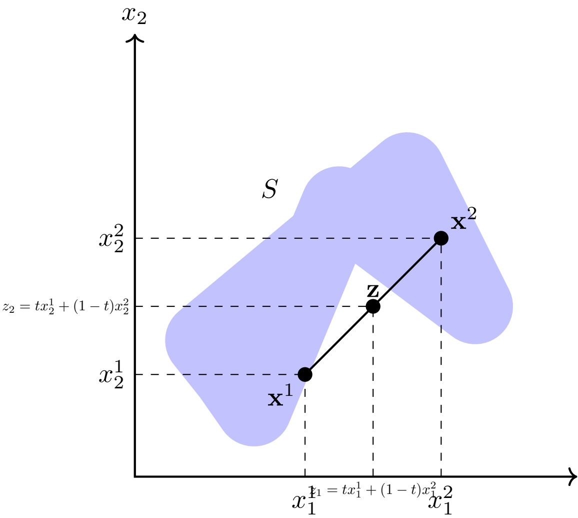
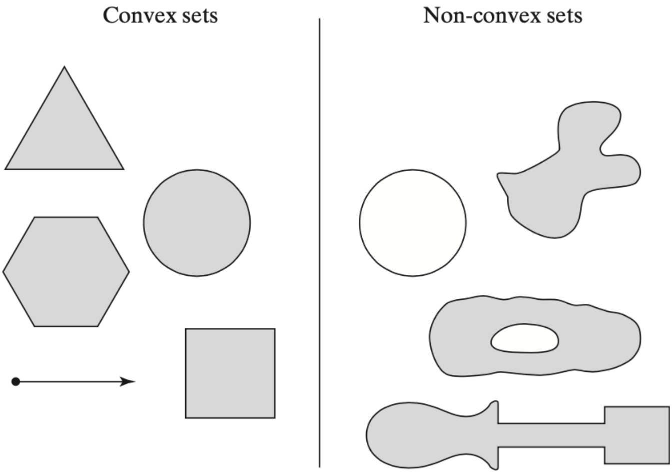


Convex sets: Main points

- Intuition: a set is convex if for any two points in the set, all weighted average of those two points (where the weights sums to 1) are also points in the same set.
- weighted average = convex combination.
- We say that $\mathbf{z}$ is a convex combination of $\mathbf{x}^1$ and $\mathbf{x}^2$ if $\mathbf{z} = t\mathbf{x}^1 + (1-t)\mathbf{x}^2$ for some number t between zero and 1.
- A convex combination $\mathbf{z}$ is thus a point that, in some sense, lies between the two points $\mathbf{x}^1$ and $\mathbf{x}^2$.
- Convex sets are all "nicely behaved". They have no holes, no breaks, and no awkward curvatures on their boundaries. They are nice sets!

Convex sets: More examples



Open and closed sets

- Continuity is another important concept in terms of preferences.
- Idea: if a bundle $\mathbf{x}$ is preferred to bundle $\mathbf{y}$, then a bundle close to $\mathbf{x}$ are preferred to bundles close to $\mathbf{y}$
- How can we define points to be near each other?
- We need to understand the concept of open and closed sets.
- ... and the concepts of closed and open balls.

Open and closed ϵ-Balls

Definition (ϵ-Balls)

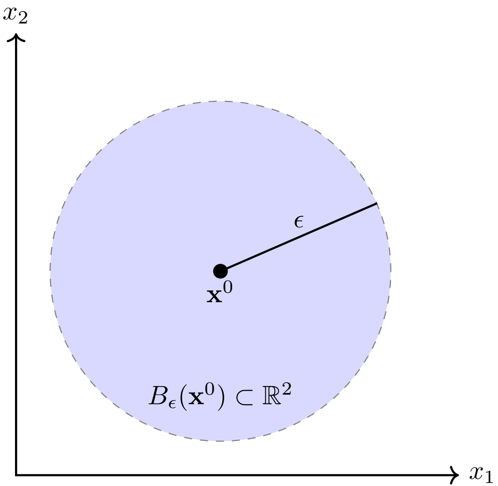
1. (Open ball) The open ϵ-ball with center $\mathbf{x}^0$ and radius $\epsilon > 0$ (a real number) is the subset of points in $\mathbb{R}^n$:

$$B_\epsilon(\mathbf{x}^0) \equiv \left\{ \mathbf{x} \in \mathbb{R}^n | \underbrace{d(\mathbf{x}^0, \mathbf{x}) < \epsilon}_{\text{strictly less than}} \right\} \tag{2}$$

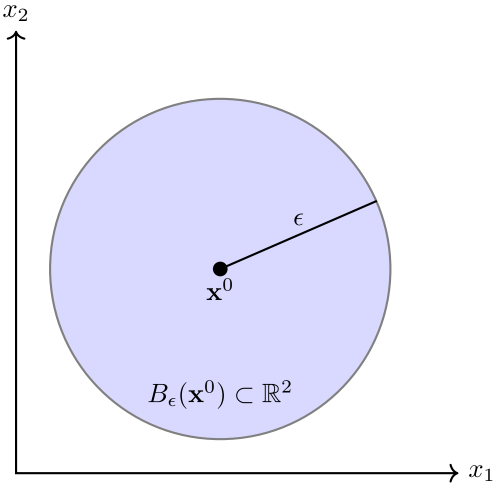
2. (Closed ball) The closed ϵ-ball with center $\mathbf{x}^0$ and radius $\epsilon > 0$ (a real number) is the subset of points in $\mathbb{R}^n$:

$$B^*_\epsilon(\mathbf{x}^0) \equiv \left\{ \mathbf{x} \in \mathbb{R}^n | \underbrace{d(\mathbf{x}^0, \mathbf{x}) \leq \epsilon}_{\text{less than or equal to}} \right\} \tag{3}$$

Open Ball



Closed ball



Open and closed sets

Definition (Open sets in $\mathbb{R}^n$)

$S \subset \mathbb{R}^n$ is an open set if, for all $\mathbf{x} \in S$, there exists some $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset S$

A set is open if around any point in it we can draw some open ball–no matter how small its radius may have to be–so that all the points in that ball will lie entirely in the set.

Definition (Closed sets in $\mathbb{R}^n$)

S is a closed set if its complement, S^c, is an open set.

An open ball is an open set

- Let S be an open ball with centre $\mathbf{x}^0$ and radius ϵ

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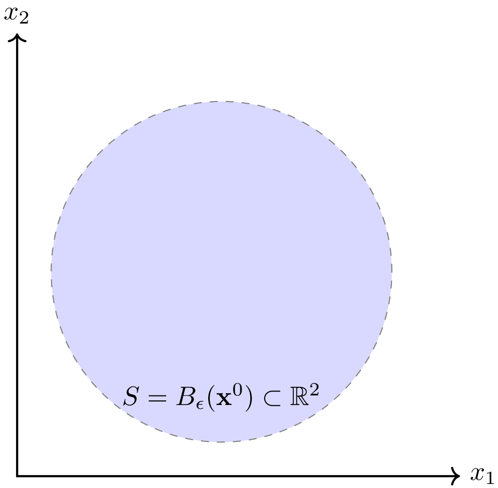

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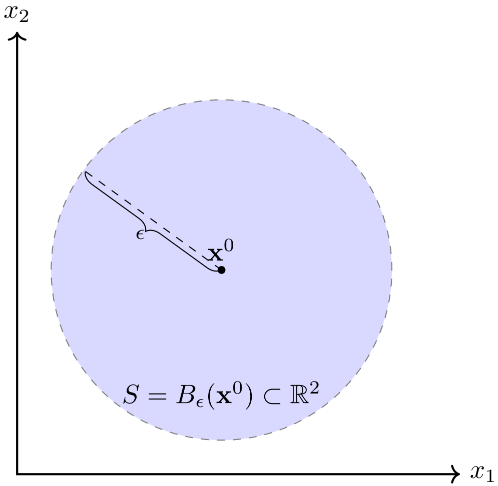
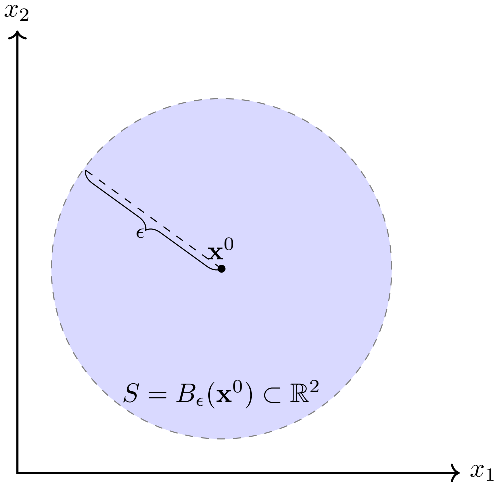


An open ball is an open set

- Let S be an open ball with centre $\mathbf{x}^0$ and radius ϵ
- It will always be possible to draw an open ball around any point $\mathbf{x} \in S$ whose points all lie withing S by choosing ϵ' carefully enough.



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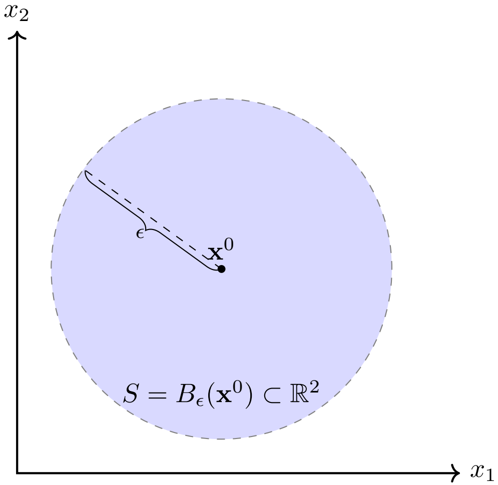
- Take any point $\mathbf{x} \in S$



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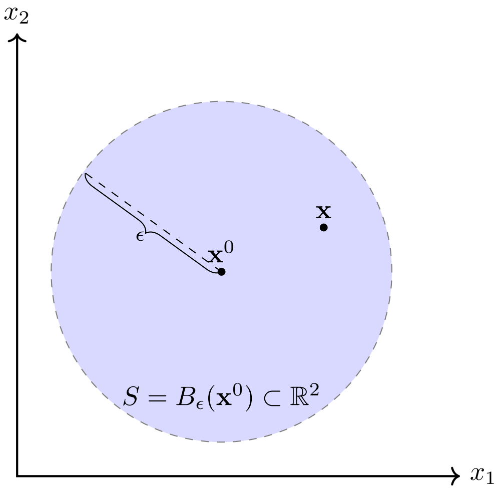

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- Take any point $\mathbf{x} \in S$
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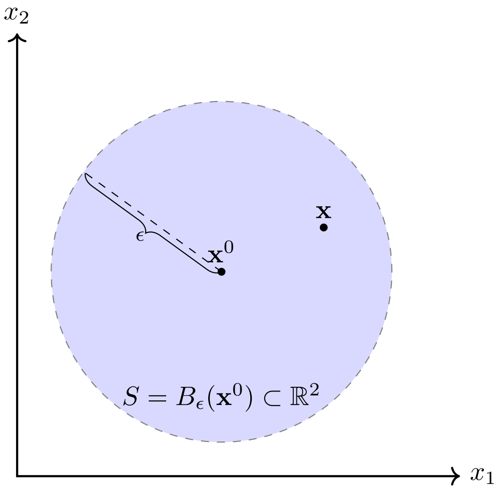

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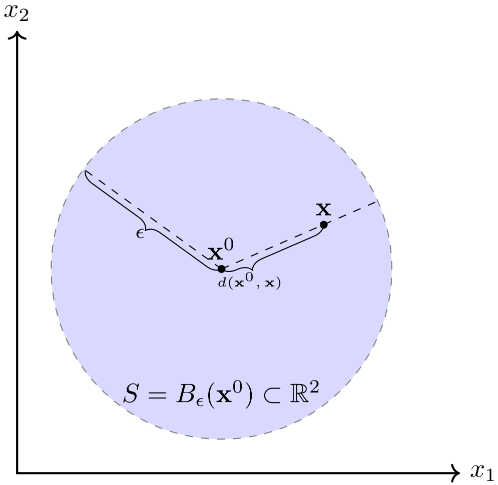

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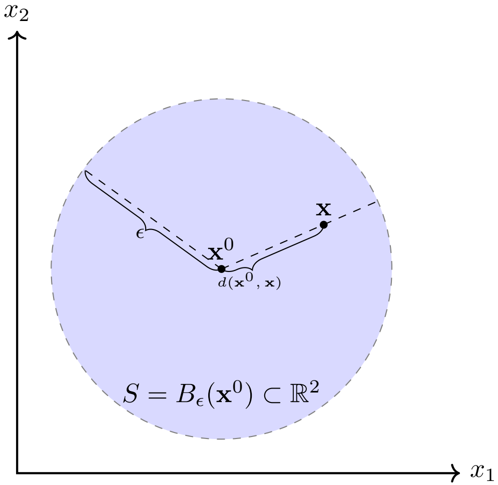

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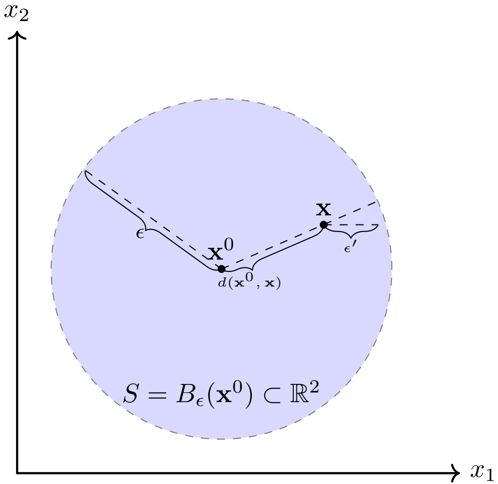

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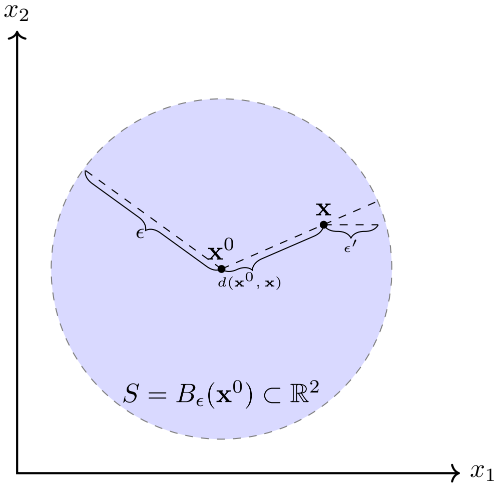
- Let $\epsilon' = \epsilon - d(\mathbf{x}^0, \mathbf{x}) > 0$
- It will be always the case that $B_{\epsilon'} \subset S$.



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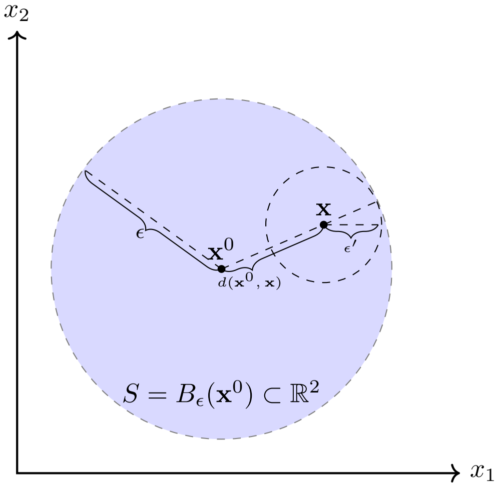
- Take any point $\mathbf{x} \in S$
- We know that $d(\mathbf{x}^0, \mathbf{x}) < \epsilon$. Thus $\epsilon - d(\mathbf{x}^0, \mathbf{x}) > 0$
- Let $\epsilon' = \epsilon - d(\mathbf{x}^0, \mathbf{x}) > 0$
- It will be always the case that $B_{\epsilon'} \subset S$.



Theorem on open sets

Theorem (On open sets in $\mathbb{R}^n$)

1. *The empty set, $\emptyset$, is an open set.*
2. *The entire space, $\mathbb{R}^n$, is an open set.*
3. *The union of open sets is an open set.*
4. *The intersection of any finite number of open sets is an open set.*

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4. *The intersection of closed sets is a closed set.*

Primitive Notions

Let's go back to the consumption set!

Primitive Notions

Assumption: Properties of the consumption set, X

The minimal requirements on the consumption set are:

1. $\emptyset \neq X \subseteq \mathbb{R}^n_+$
2. X is closed.
3. X is convex.
4. $\mathbf{0} \in X$

- **Feasible set**: $\mathcal{B} \subseteq X$. All those alternatives consumptions plans that are both conceivable and realistically obtainable given the consumer's circumstances.

Preferences

- Consumer preferences are axiomatic.
- The rest of the theory then builds logically from these axioms.
- Axioms of consumer choice: formal mathematical expression to fundamental aspects of consumer behavior and attitudes toward the objects of choice.

Binary relations

- We represent the consumer's preferences by a binary relation $\succcurlyeq$:

If $\mathbf{x}^1 \succcurlyeq \mathbf{x}^2$, we say that "$\mathbf{x}^1$ is at least as good as $\mathbf{x}^2$", for this consumer.

- Do not confuse $\succcurlyeq$ with $\geq$!

Completeness and transitivity

Axiom 1: Completeness.

For all $\mathbf{x}^1$ and $\mathbf{x}^2$ in X, either $\mathbf{x}^1 \succcurlyeq \mathbf{x}^2$ or $\mathbf{x}^2 \succcurlyeq \mathbf{x}^1$

Axiom 2: Transitivity.

For any three elements $\mathbf{x}^1, \mathbf{x}^2$ and $\mathbf{x}^3$ in X, if $\mathbf{x}^1 \succcurlyeq \mathbf{x}^2$ and $\mathbf{x}^2 \succcurlyeq \mathbf{x}^3$, then $\mathbf{x}^1 \succcurlyeq \mathbf{x}^3$

- Axiom 1: Consumers are able to express preference or indifference $\Longrightarrow$ no holes in the ordering of preferences.
- These two axioms together imply that the consumer can completely rank any finite number of elements in the consumption set, X, from best to worst, possible with some ties.

Comments on completeness and transitivity

- Are you always able to order your preferred bundles?
- Transitivity: the heart of rationality.
- If you comply with completeness and transitivity, then you are rational!

Comments on completeness and transitivity

Example

Imagine that you have to choose among 3 CDs: 1) Nirvana 2) Don Omar 3) Ricardo Arjona. Now consider the following question

1. Which is better, Nirvana or Don Omar.

Comments on completeness and transitivity

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- Were you able to respond all the questions? Congratulations, your preferences are complete.

Comments on completeness and transitivity

Example

Imagine that you have to choose among 3 CDs: 1) Nirvana 2) Don Omar 3) Ricardo Arjona. Now consider the following question

1. Which is better, Nirvana or Don Omar.
2. Which is better, Nirvana or Ricardo Arjona.
3. Which is better, Don Omar or Ricardo Arjona.

- Were you able to respond all the questions? Congratulations, your preferences are complete.
- Did you preferences creates a cycle. Congratulations, your preferences are transitive.

When transitivity might fail?

- Perceptible differences: choose between two very similar shades of gray for painting the room.
- Framing problem (or cognitive bias): people react to a particular choice in different ways depending on how it is presented; e.g. as a loss or as a gain. (Prospect Theory)

Preference relation

Definition (Preference Relation)

The binary relation $\succsim$ on the consumption set X is called a preference relation if it satisfies Axioms 1 (completeness) and 2 (transitivity).

Definition (Strict preference relation)

The binary relation $\succ$ on the consumption set X is defined as follows:

$$\mathbf{x}^1 \succ \mathbf{x}^2 \quad \text{iff} \quad \mathbf{x}^1 \succsim \mathbf{x}^2 \quad \text{and} \quad \mathbf{x}^2 \not\succsim \mathbf{x}^1$$

The relation $\succ$ is called the strict preference relation induced by $\succsim$, or simply the strict preference relation when $\succsim$ is clear. The phrase $\mathbf{x}^1 \succ \mathbf{x}^2$ is read, "$\mathbf{x}^1$ is strictly preferred to $\mathbf{x}^2$."

We can also write:

$$(\mathbf{x}^1 \succ \mathbf{x}^2) \quad \Longleftrightarrow \quad (\mathbf{x}^1 \succsim \mathbf{x}^2) \quad \wedge \neg \quad (\mathbf{x}^2 \succsim \mathbf{x}^1)$$

where $\wedge$ is the logical operator for AND, $\neg$ is the logical operator for NOT.

Preference relation

Definition (Indifference Relation)

The binary relation $\sim$ on the consumption set X is defined as follows:

$$\mathbf{x}^1 \sim \mathbf{x}^2 \quad \text{iff} \quad \mathbf{x}^1 \succcurlyeq \mathbf{x}^2 \quad \text{and} \quad \mathbf{x}^2 \succcurlyeq \mathbf{x}^1$$

The relation $\sim$ is called the indifference relation induced by $\succcurlyeq$, or simply the indifference relation when $\succcurlyeq$ is clear. The phrase $\mathbf{x}^1 \sim \mathbf{x}^2$ is read, "$\mathbf{x}^1$ is indifferent to $\mathbf{x}^2$."

We can also write:

$$(\mathbf{x}^1 \sim \mathbf{x}^2) \quad \Longleftrightarrow \quad (\mathbf{x}^1 \succcurlyeq \mathbf{x}^2) \quad \wedge \quad (\mathbf{x}^2 \succcurlyeq \mathbf{x}^1)$$

Example:

Show that $\forall \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in X$: $\mathbf{x}^1 \sim \mathbf{x}^2 \sim \mathbf{x}^3 \implies \mathbf{x}^1 \sim \mathbf{x}^3$.

Example:

Show that $\forall \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in X$: $\mathbf{x}^1 \sim \mathbf{x}^2 \sim \mathbf{x}^3 \implies \mathbf{x}^1 \sim \mathbf{x}^3$.

Solution: By definition of indifference

$$\begin{aligned} \mathbf{x}^1 \sim \mathbf{x}^2 &\iff \mathbf{x}^1 \succcurlyeq \mathbf{x}^2 \wedge \mathbf{x}^2 \succcurlyeq \mathbf{x}^1 \\ \mathbf{x}^2 \sim \mathbf{x}^3 &\iff \mathbf{x}^2 \succcurlyeq \mathbf{x}^3 \wedge \mathbf{x}^3 \succcurlyeq \mathbf{x}^2 \end{aligned}$$

which can be also be written as:

$$(\mathbf{x}^1 \sim \mathbf{x}^2) \wedge (\mathbf{x}^2 \sim \mathbf{x}^3) \iff \left[(\mathbf{x}^1 \succcurlyeq \mathbf{x}^2) \wedge (\mathbf{x}^2 \succcurlyeq \mathbf{x}^1)\right] \wedge \left[(\mathbf{x}^2 \succcurlyeq \mathbf{x}^3) \wedge (\mathbf{x}^3 \succcurlyeq \mathbf{x}^2)\right]$$

Taking into account logic operators,

$$\begin{aligned} (\mathbf{x}^1 \sim \mathbf{x}^2) \wedge (\mathbf{x}^2 \sim \mathbf{x}^3) &\iff \left[(\mathbf{x}^1 \succcurlyeq \mathbf{x}^2) \wedge (\mathbf{x}^2 \succcurlyeq \mathbf{x}^3)\right] \wedge \left[(\mathbf{x}^3 \succcurlyeq \mathbf{x}^2) \wedge (\mathbf{x}^2 \succcurlyeq \mathbf{x}^1)\right] \\ &\iff (\mathbf{x}^1 \succcurlyeq \mathbf{x}^3) \wedge (\mathbf{x}^3 \succcurlyeq \mathbf{x}^1) \quad \text{By transitivity} \\ &\iff \mathbf{x}^1 \sim \mathbf{x}^3 \quad \text{By definition of indifference} \end{aligned}$$

Preference relation

Example (Question)

Can we write $\neg(\mathbf{x}^1 \succcurlyeq \mathbf{x}^2) \wedge \neg(\mathbf{x}^2 \succcurlyeq \mathbf{x}^1)$?

Preferences Sets

- Recall that under Completeness, the consumer is able to compare $\mathbf{x}^0$ (or any bundle), with any other plan in X.

Definition (Sets in X derived form the preference relation)

Let $\mathbf{x}^0$ be any point in the consumption set, X. Relative to any such point, we can define the following subsets of X:

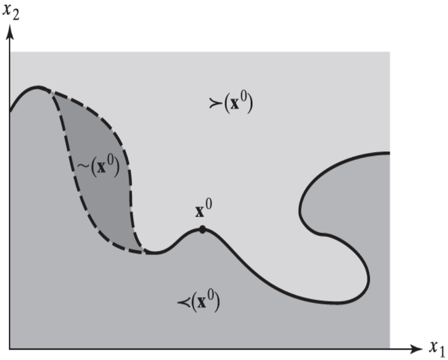
1. $\succsim(\mathbf{x}^0) \equiv \{\mathbf{x}|\mathbf{x} \in X, \mathbf{x} \succsim \mathbf{x}^0\}$, called the "at least as good as" set.
2. $\precsim(\mathbf{x}^0) \equiv \{\mathbf{x}|\mathbf{x} \in X, \mathbf{x}^0 \succsim \mathbf{x}\}$, called the "no better than" set.
3. $\prec(\mathbf{x}^0) \equiv \{\mathbf{x}|\mathbf{x} \in X, \mathbf{x}^0 \succ \mathbf{x}\}$, called the "worse than" set.
4. $\succ(\mathbf{x}^0) \equiv \{\mathbf{x}|\mathbf{x} \in X, \mathbf{x} \succ \mathbf{x}^0\}$, called the "preferred to" set.
5. $\sim(\mathbf{x}^0) \equiv \{\mathbf{x}|\mathbf{x} \in X, \mathbf{x} \sim \mathbf{x}^0\}$, called the "indifference" set.

- Note that the sets $\prec(\mathbf{x}^0)$, $\succ(\mathbf{x}^0)$, and $\sim(\mathbf{x}^0)$ are mutually exclusive.

Axioms of regularity

How do look like preferences satisfying completeness and transitivity?

- $\mathbf{x}^0 = (x_1^0, x_2^0)$



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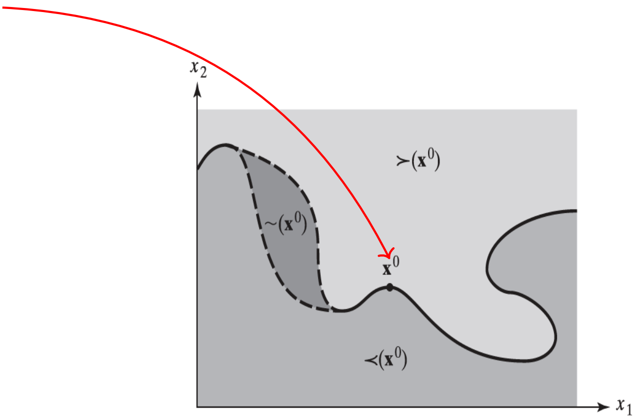
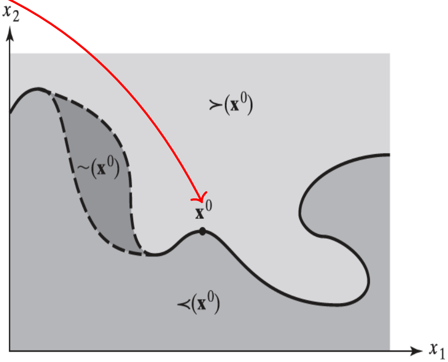


- $\mathbf{x}^0 = (x_1^0, x_2^0)$
- Points that are worse than $\mathbf{x}^0$



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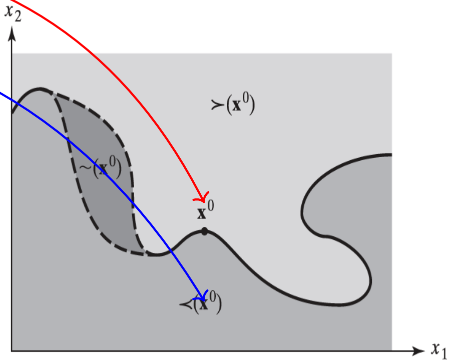
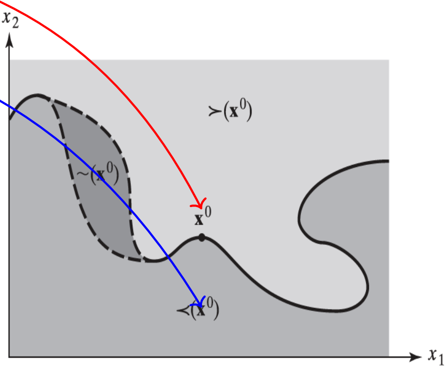


- $\mathbf{x}^0 = (x_1^0, x_2^0)$
- Points that are worse than $\mathbf{x}^0$
- Points that are indifferent to $\mathbf{x}^0$



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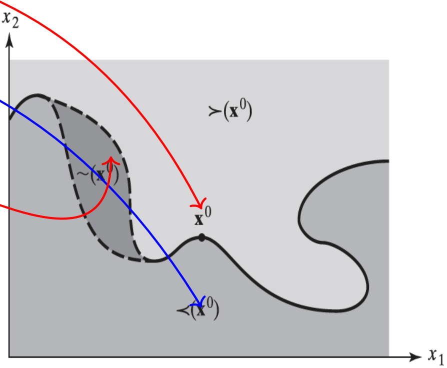
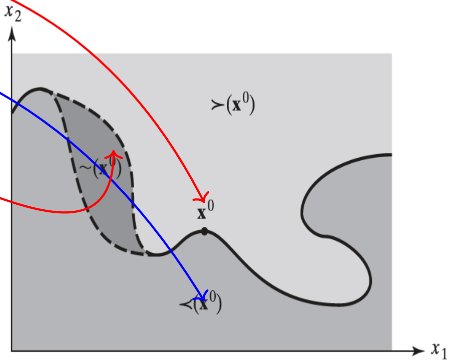


- $\mathbf{x}^0 = (x_1^0, x_2^0)$
- Points that are worse than $\mathbf{x}^0$
- Points that are indifferent to $\mathbf{x}^0$
- Points that are better than $\mathbf{x}^0$



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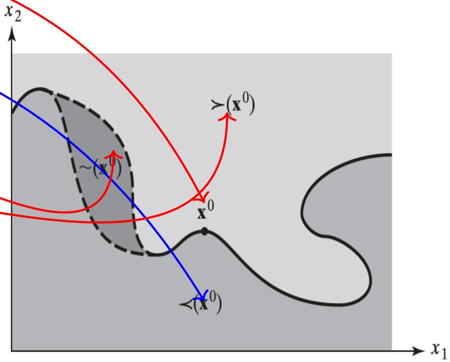

- $\mathbf{x}^0 = (x_1^0, x_2^0)$
- Points that are worse than $\mathbf{x}^0$
- Points that are indifferent to $\mathbf{x}^0$
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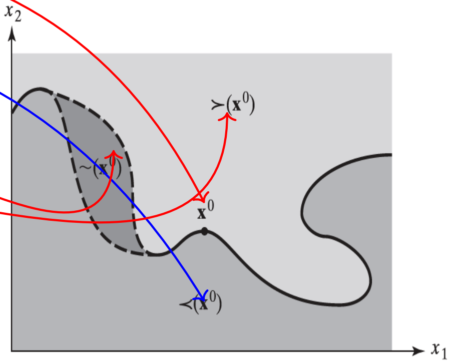
- For any bundle $\mathbf{x}^0$ the three sets $\succ(\mathbf{x}^0)$, $\sim(\mathbf{x}^0)$, and $\prec(\mathbf{x}^0)$ partition the consumption set.



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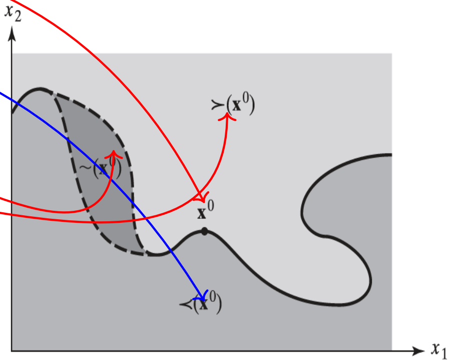
- Note that $\succcurlyeq(\mathbf{x}^0) \cap \preccurlyeq(\mathbf{x}^0) = \sim(\mathbf{x}^0)$



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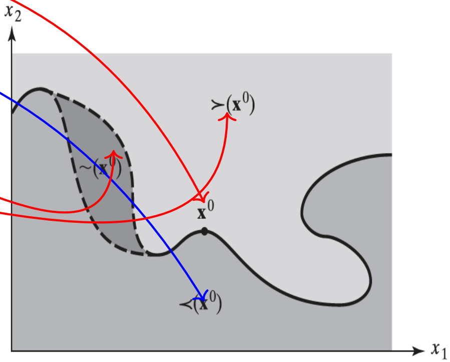
- Note that $\succsim(\mathbf{x}^0) \cap \precsim(\mathbf{x}^0) = \sim(\mathbf{x}^0)$
- We need other mathematical regularities.



Continuity

Note that a bundle in the boundary of $\sim (\mathbf{x})$ does not belong to any set.

Intuition

Continuity is a property that captures the idea that if a bundle $\mathbf{x}$ is preferred to bundle $\mathbf{y}$, then a bundle close to $\mathbf{x}$ are preferred to bundles close to $\mathbf{y}$

Continuity

The first is an axiom whose only effect is to impose a kind of topological regularity on preferences, and whose primary contribution will be clear a bit later. From now on we explicitly set $X = \mathbb{R}^n_+$

Axiom 3: Continuity

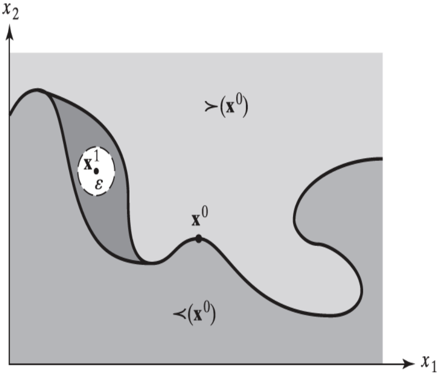
For all $\mathbf{x} \in X = \mathbb{R}^n_+$, the "at least as good as" set, $\succcurlyeq(\mathbf{x})$, and the "no better than" set $\preccurlyeq(\mathbf{x})$, are closed in $\mathbb{R}^n_+$

Consequences of continuity

- To say that $\succcurlyeq(\mathbf{x})$ is closed in $\mathbb{R}^n_+$ is to say that its complement, $\prec(\mathbf{x})$, is open in $\mathbb{R}^n_+$
- The "at least as good as" set, $\succcurlyeq(\mathbf{x})$, and the "no better than" set $\preccurlyeq(\mathbf{x})$ include their boundaries.
- Continuity is the same as the following: if each element $\mathbf{y}^n$ of a sequence of bundles is at least as good as (no better than) $\mathbf{x}$, and $\mathbf{y}^n \to \mathbf{y}$, then $\mathbf{y}$ is at least as good as (no better than) $\mathbf{x}$
- Note that because $\succcurlyeq(\mathbf{x})$ and $\preccurlyeq(\mathbf{x})$ are closed, so, too is $\sim(\mathbf{x})$ because the latter is the intersection of the former two.
- Example of discontinuity: Lexicographic preferences.

Hypothetical preferences satisfying completeness, transitivity and continuity

- Closed Set



Hypothetical preferences satisfying completeness, transitivity and continuity

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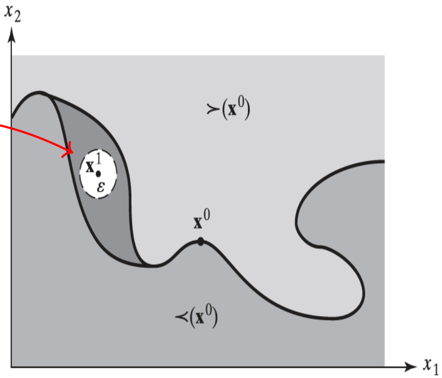
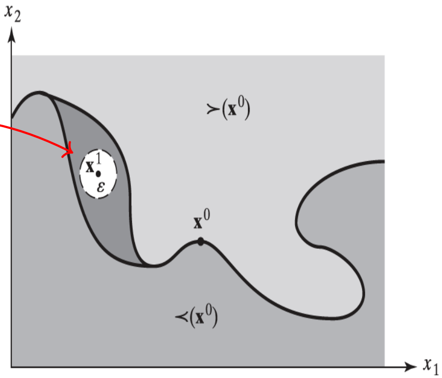


Hypothetical preferences satisfying completeness, transitivity and continuity

- Closed Set
- More is always better than less?



Local non-satiation

Intuition

Within a **vicinity** of a given point $\mathbf{x}^0$, no mater how small that vicinity is, there will always be at least one point $\mathbf{x}$ that the consumer prefers to $\mathbf{x}^0$.

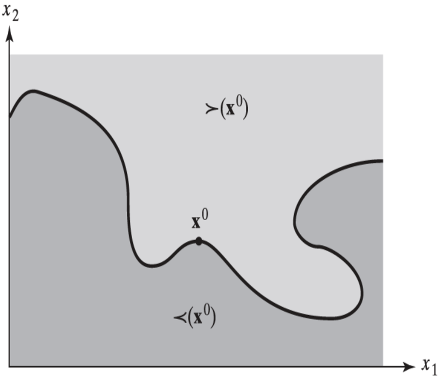
Axiom 4': Local non-satiation

For all $\mathbf{x}^0 \in \mathbb{R}^n_+$, and for all $\epsilon > 0$, there exists some $\mathbf{x} \in \mathcal{B}_\epsilon(\mathbf{x}^0) \cap \mathbb{R}^n_+$ such that $\mathbf{x} \succ \mathbf{x}^0$

We preclude the possibility that the consumer can even imagine having all his wants and whims for commodities completely satisfied. (Rules out thickness of indifference curve)

Hypothetical preferences satisfying completeness, transitivity, continuity and local non-satiation

- LNS does not rule out the possibility that the preferred alternative may involve less of some or even all commodities.
- It does not imply that giving the consumer more of everything necessarily makes that consumer better off.
- It says that we can always find a better bundle in terms of sets, no in terms of quantities of the goods.



Strict Monotonicity

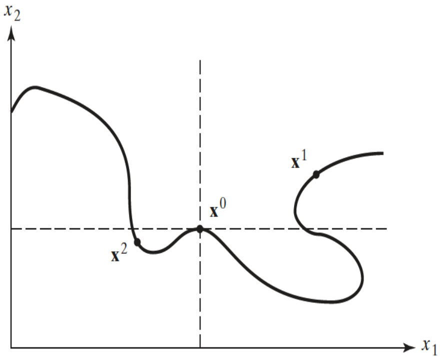
Axiom 4: Strict Monotonicity

$\forall \mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$ if $\mathbf{x}^0 \geq \mathbf{x}^1$, then $\mathbf{x}^0 \succsim \mathbf{x}^1$, while if $\mathbf{x}^0 \gg \mathbf{x}^1 \implies \mathbf{x}^0 \succ \mathbf{x}^1$

- $\mathbf{x}^0 \geq \mathbf{x}^1$ implies that $\mathbf{x}^0$ contains at least as much of every good as does $\mathbf{x}^1$.
- $\mathbf{x}^0 \gg \mathbf{x}^1$ implies that $\mathbf{x}^0$ contains strictly more of every good that $\mathbf{x}^1$
- Note: Axiom 4 $\implies$ Axiom 4'

Hypothetical preferences satisfying completeness, transitivity and continuity

- We don't want 'bend upward' indifference curve



Hypothetical preferences satisfying completeness, transitivity and continuity

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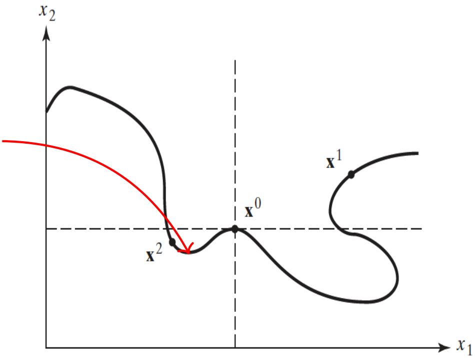
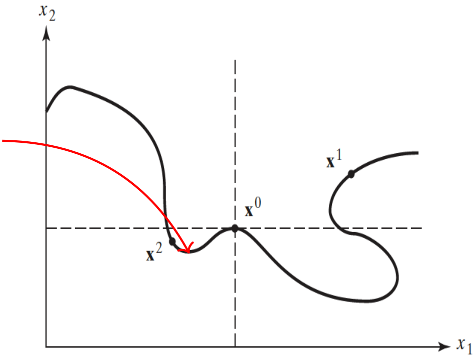


Hypothetical preferences satisfying completeness, transitivity and continuity

- We don't want 'bend upward' indifference curve
- Quadrant I: points strictly preferred to $\mathbf{x}^0$



Hypothetical preferences satisfying completeness, transitivity and continuity

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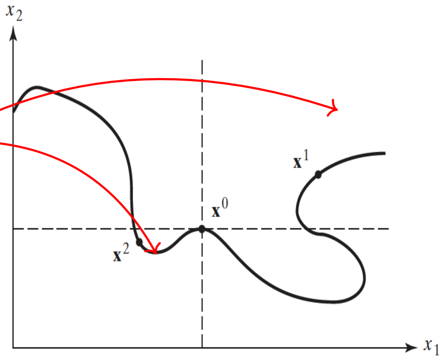
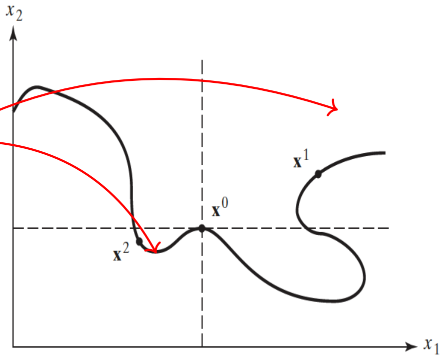


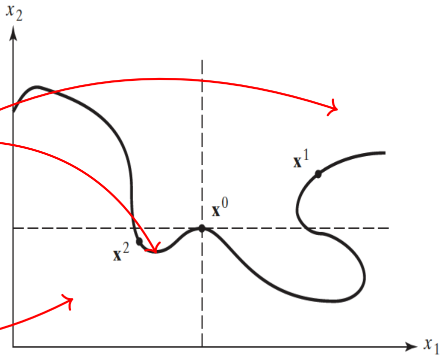
Hypothetical preferences satisfying completeness, transitivity and continuity

- We don't want 'bend upward' indifference curve
- Quadrant I: points strictly preferred to $\mathbf{x}^0$
- Quadrant III: Less of both goods



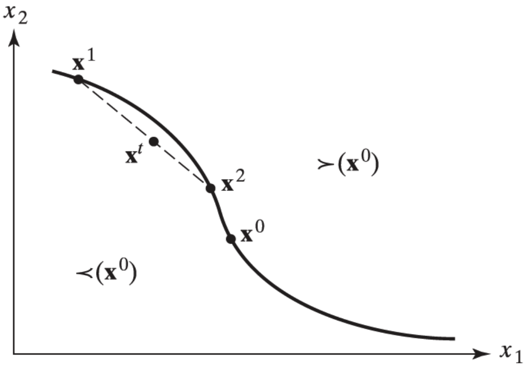
Hypothetical preferences satisfying completeness, transitivity and continuity

- We don't want 'bend upward' indifference curve
- Quadrant I: points strictly preferred to $\mathbf{x}^0$
- Quadrant III: Less of both goods



Consequences of strict monotonicity

Figure: Hypothetical preferences satisfying completeness, transitivity, continuity and strict Monotonicity



Convexity on preferences

Axiom 5': Convexity

If $\mathbf{x}^0 \succsim \mathbf{x}^1$, then $\theta\mathbf{x}^1 + (1-\theta)\mathbf{x}^0 \succsim \mathbf{x}^0$ for all $\theta \in [0,1]$.

A slightly stronger version of this is the following

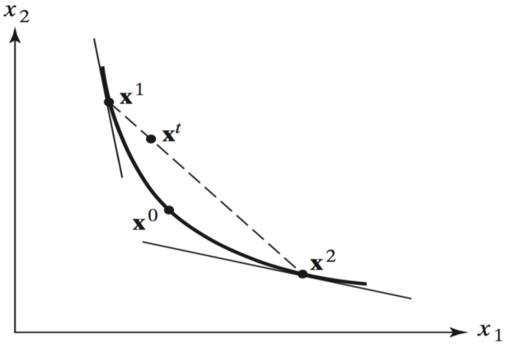
Axiom 5: Strict Convexity

If $\mathbf{x}^1 \neq \mathbf{x}^0$ and $\mathbf{x}^0 \succsim \mathbf{x}^1$, then $\theta\mathbf{x}^1 + (1-\theta)\mathbf{x}^0 \succ \mathbf{x}^0$ for all $\theta \in (0,1)$.

Note: $X = \mathbb{R}^n_+$ is convex. Then $\theta\mathbf{x}^1 + (1-\theta)\mathbf{x}^0 \in X$.

Consequences of Convexity

Figure: Hypothetical preferences satisfying completeness, transitivity, continuity, strict Monotonicity, and convexity



Consequences of Convexity

- Averages are preferred to extreme consumption.
- Marginal Rate of Substitution: principle of diminishing marginal rate of substitution in consumption.

The utility function

- The utility function is simply a convenient device for summarizing the information contained in the consumer's preference relation (no more and no less)
- With the utility function we'll be able to use calculus methods.
- In the modern theory, the preference relation is taken to be the primitive, most fundamental characterization of preferences.

The utility function

Example

Suppose choice set is $X = \{a, b, c\}$ and that $a \succ b \succ c$. Now, define a utility function over X as follows

$$u(a) = 3, u(b) = 2, u(c) = 1$$

This function is said to represent the preference ordering $a \succ b \succ c$. We interpret a higher utility index with a better choice. Note that the choice of index is completely arbitrary. For example

$$\begin{aligned} u(a) &= 100 \\ u(b) &= \sqrt{2} \\ u(c) &= 1 \end{aligned}$$

represent exactly the same preferences.

The utility function

Example

"I prefer the taller soccer player" can be expressed formally by: X is the set of all conceivable basketball players, and $u(x)$ is the height of player x.

The utility function

Definition (A utility function representing the preference relation $\succsim$)

A real-valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is called a utility function representing the preference relation $\succsim$, if $\forall \mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$,

$$u(\mathbf{x}^0) \geq u(\mathbf{x}^1) \iff \mathbf{x}^0 \succsim \mathbf{x}^1$$

The utility function

- u represents a consumer's preference relation if it assigns higher numbers to preferred bundles.
- Existence of u: it can be shown that any binary relation that is complete, transitive and continuous can be represented by a continuous real-valued utility function.

The utility function

Theorem (Existence of utility function)

If the binary relation $\succcurlyeq$ is complete, transitive, continuous, and strictly monotonic, there exists a continuous real-valued, $u : \mathbb{R}^n_+ \rightarrow \mathbb{R}$, which represents $\succcurlyeq$

- It is only an existence theorem, it might be more than one function that represents the same preferences.
- If we can dream up just one function that is continuous and that represents the given preferences, we will have proved the theorem

The utility function

- So, we can represent preferences either in terms of the primitive set-theoretic preference relation or in terms of a numerical representation, a continuous utility function.
- Utility representation is never unique: u, $v = u + 5$, $v = u^3$ (all ranks bundles the same way u does)
- They convey ordinal information, no more no less.
- No significance whatsoever can be attached to the actual numbers assigned by a given utility function to particular bundles.

The utility function

Theorem (Invariance of the utility function to positive monotonic transforms)

Let $\succcurlyeq$ be a preference relation on $\mathbb{R}^n_+$ and suppose $u(\mathbf{x})$ is a utility function that represents it. Then $v(\mathbf{x})$ also represents $\succcurlyeq$ iff $v(\mathbf{x}) = f(u(\mathbf{x}))$ for every $\mathbf{x}$, where $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing on the set of values taken on by u.

The utility function

- Monotonic Transformations:
  - Multiplication by a positive number.
  - Adding a positive number.
  - Taking the $\ln(\cdot)$
  - Exponentiation.
  - Raising to an odd power.
  - Raising to an even power for functions defined over non-negative values.

Quasiconcavity

Why quasiconcavity?

- Concavity is a strong restriction to place on a function.
- We want to impose only the weakest possible restriction needed to guarantee the result sought.

Quasiconcavity

Definition (Quasi-Concave Functions)

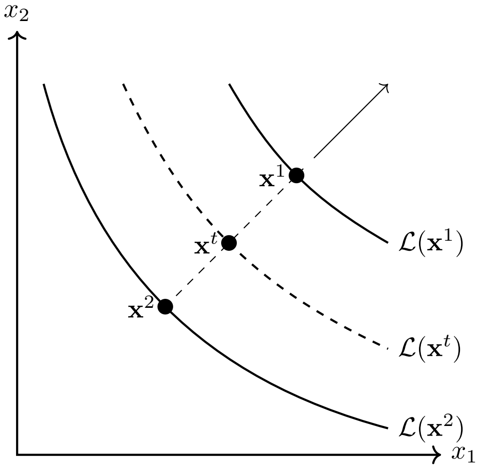
Let $A \subset \mathbb{R}^n$ be a convex set, and f a real-valued function on A. Then f is qusi-concave on A if and only if either of the following equivalent conditions is satisfied for all $\mathbf{x}^1$ and $\mathbf{x}^2$ in A and all $t \in [0,1]$:

$$f(\mathbf{x}^1) \geq f(\mathbf{x}^2) \implies f\left[t\mathbf{x}^1 + (1-t)\mathbf{x}^2\right] \geq f(\mathbf{x}^2) \tag{4}$$

$$f\left[t\mathbf{x}^1 + (1-\theta)\mathbf{x}^2\right] \geq \min\left[f(\mathbf{x}^1), f(\mathbf{x}^2)\right] \tag{5}$$

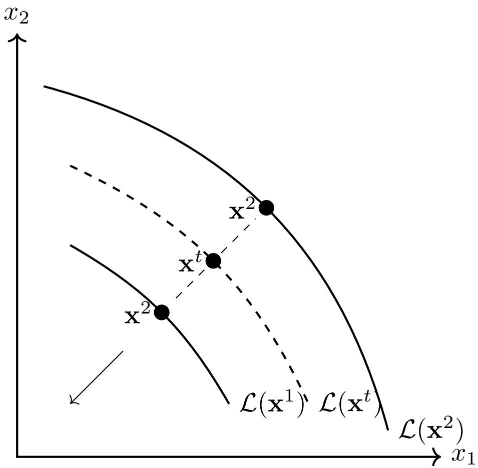
Quasiconcavity and increasing function

- Let $y = f(x_1, x_2)$.
- Pick any two points $\mathbf{x}^1$ and $\mathbf{x}^2$ in its domain.
- We assume that $f(\mathbf{x}^1) \geq f(\mathbf{x}^2)$
- Form $\mathbf{x}^t$.



Quasiconcavity and decreasing function

- Let $y = f(x_1, x_2)$.
- Pick any two points $\mathbf{x}^1$ and $\mathbf{x}^2$ in its domain.
- We assume that $f(\mathbf{x}^1) \leq f(\mathbf{x}^2)$
- Form $\mathbf{x}^t$.



Quasiconcavity

Theorem (Quasi-Concavity and Convex set)

Suppose A is a convex subset of $\mathbb{R}^n$, and f is a real-valued function on A. Then, f is quasi-concave on A if and only if for every $\alpha \in \mathbb{R}$, the set:

$$S(\alpha) = \{\mathbf{x} \in A | f(\mathbf{x}) \geq \alpha\}$$

is a convex set in $\mathbb{R}^n$

Then $S(\alpha)$ is called an **upper level set** for f. It consists of those points in A that give values of f that are greater than or equal to α

The utility function

Theorem (Properties of Preferences and Utility Functions)

Let $\succcurlyeq$ be represented by $u : \mathbb{R}^n_+ \to \mathbb{R}$. Then:

1. $u(\mathbf{x})$ *is strictly increasing if and only if* $\succcurlyeq$ *is strictly monotonic.*
2. $u(\mathbf{x})$ *is quasiconcave if and only if* $\succcurlyeq$ *is convex.*
3. $u(\mathbf{x})$ *is strictly quasiconcave iff* $\succcurlyeq$ *is strictly convex*

- A preference relation $\succcurlyeq$ is convex $\iff$ the upper preference sets $\succcurlyeq(\mathbf{y})$ are convex for every $\mathbf{y}$.
- A utility function u represents a convex preference relation iff u is quasiconcave.

We will assume that the utility representation is differentiable whenever necessary.

Preferences and Utility Function

- $\succcurlyeq$ on X is locally nonsatiated $\iff$ for every $\mathbf{x} \in X$ and $\epsilon > 0$, there exists $y \in X$ such that $|\mathbf{y} - \mathbf{x}| < \epsilon$ and $u(\mathbf{y}) > u(\mathbf{x})$.
- $\succcurlyeq$ on X is monotone (strict monotone) $\iff$ $\mathbf{x} \gg \mathbf{y}$ ($\mathbf{x} > \mathbf{y}$) implies $u(x) > u(y)$ for any $\mathbf{x}, \mathbf{y} \in X$.
- $\succcurlyeq$ on X is convex $\iff$ u is quasi-concave, i.e. $u(\mathbf{y}) > u(\mathbf{x})$ and $u(\mathbf{z}) \geq u(\mathbf{x})$ imply $u(\theta\mathbf{y} + (1-\theta)\mathbf{z}) \geq u(x)$ for any $\theta \in [0, 1]$.
- $\succcurlyeq$ on X is strictly convex $\iff$ u is strictly quasi-concave, i.e. $u(\mathbf{y}) > u(\mathbf{x})$ and $u(\mathbf{z}) \geq u(\mathbf{x})$ with $\mathbf{y} \neq \mathbf{z}$ imply $u(\theta\mathbf{y} + (1-\theta)\mathbf{z}) > u(x)$ for any $\theta \in (0, 1)$.

Marginal rate of substitution

- The amount of good 1 that the consumer is willing to give up for 1 unit of good 2, or the internal rate of trade between the two goods.
- Dimishing marginal rate of substitution occurs when, the more good 2 you have, the less good 1 you are willing to give up

$$MRS_{ij}(\mathbf{x}) \equiv \frac{\partial u(\mathbf{x})/\partial x_i}{\partial u(\mathbf{x})/\partial x_j}$$