

Lecture 2: Spatial Models



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 - Spatial Lag Model
 - Reduced Form and Parameter Space
 - Expected Value and Variance
 - Spatial Durbin Model
 - Spatial Error Model
 - SAC

- 2 Motivating Spatial Models
 - SLM as a Long-run Equilibrium
 - SEM and Omitted Variables Motivation
 - SDM and Omitted Variables Motivation

- 3 Interpreting Spatial Models
 - Measuring Spillovers
 - Partitioning Global Effects Estimates Over Space
 - Lesage's Book Example

- 4 Empirical Example: Kirby and Lesage, 2009.
 - Context
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Goals

Goals

- To understand how to model spatial dependency in a regression context.
- To understand the taxonomy of different spatial models.
- To motivate each spatial model.
- To derive and interpret spatial spillover effects.

Reading for: Spatial Models

- (LK)-Sections 1.2 to the end and full chapter 2
- Elhorst, J. P. (2010). Applied spatial econometrics: raising the bar. *Spatial Economic Analysis*, 5(1), 9-28.
- LeSage, J. P. (2014). What Regional Scientists Need to Know about Spatial Econometrics. *The Review of Regional Studies*, 44(1):13-32.
- Kirby, D. K., & LeSage, J. P. (2009). Changes in commuting to work times over the 1990 to 2000 period. *Regional Science and Urban Economics*, 39(4), 460-471.
- Golgher, A. B., & Voss, P. R. (2016). How to interpret the coefficients of spatial models: Spillovers, direct and indirect effects. *Spatial Demography*, 4(3), 175-205.

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Spatial Lag Model (SLM)

Recall our degree-of-freedom problem: It is not possible to model spatial dependency using traditional econometric

.

Spatial Lag Model (SLM)

Consider:

$$y_i = \alpha + \rho \sum_{j=1}^n w_{ij} y_j + \epsilon_i, \quad i = 1, \dots, n, \quad (1)$$

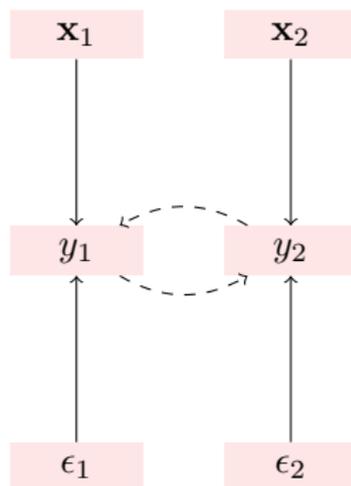
where:

- w_{ij} th element of \mathbf{W} ,
- $\sum_{j=1}^n w_{ij} y_j$ is the weighted average of the dependent variable,
- ϵ_i is the error term such that $\mathbb{E}(\epsilon_i) = 0$.
- ρ is the spatial autoregressive parameter which measures the **intensity of the spatial interdependency**.
 - $\rho > 0$: positive spatial dependence,
 - $\rho < 0$: negative spatial dependence.
 - $\rho = 0$: traditional OLS model.

This model is known as the **pure SLM** or **the Spatial Autoregressive Regression (SAR)**.

Spatial Lag Model (SLM)

Figure: The SLM for Two Regions



Effects

\rightarrow : non-spatial effects
 $--\rightarrow$: spatial effects

Spatial Lag Model (SLM)

A full SLM specification with covariates in matrix form can be written as:

$$\mathbf{y} = \alpha \mathbf{1}_n + \rho \mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad (2)$$

where \mathbf{y} is an $n \times 1$ vector of observations on the dependent variable, \mathbf{X} is an $n \times K$ matrix of observations on the explanatory variables, $\boldsymbol{\beta}$ is the $K \times 1$ vector of parameters, and $\mathbf{1}_n$ is a $n \times 1$ vector of ones.

It is also important to find the **reduced form** of the process.

Reduce Form

The **reduced form** of a system of equations is the result of solving the system for the **endogenous variables**. This gives the latter as functions of the exogenous variables, if any. For example, the general expression of a structural form is $f(\mathbf{y}, \mathbf{X}, \varepsilon) = \mathbf{0}$, whereas the reduced form of this model is given by $\mathbf{y} = g(\mathbf{X}, \varepsilon)$, with g as function.

Reduce Form

$$\begin{aligned} \mathbf{y} &= \rho \mathbf{W} \mathbf{y} + \mathbf{X} \boldsymbol{\beta} + \boldsymbol{\varepsilon} \\ \mathbf{y} &= (\mathbf{I}_n - \rho \mathbf{W})^{-1} \mathbf{X} \boldsymbol{\beta} + (\mathbf{I}_n - \rho \mathbf{W})^{-1} \boldsymbol{\varepsilon} \end{aligned}$$

- We need $(\mathbf{I}_n - \rho \mathbf{W})$ to be **invertible**.
- From standard algebra theory: \mathbf{A} is invertible if $\det(\mathbf{A}) \neq 0$

Reduce Form

Theorem (Invertibility)

Let \mathbf{W} be a weighting matrix, such that $w_{ii} = 0$ for all $i = 1, \dots, n$, and assume that all of the roots of \mathbf{W} are real. Assume that \mathbf{W} is not row normalized. Let ω_{min} and ω_{max} be the minimum and maximum eigen value of \mathbf{W} . Assume also that $\omega_{max} > 0$ and $\omega_{min} < 0$. Then $(\mathbf{I}_n - \rho\mathbf{W})$ is nonsingular for all:

$$\omega_{min}^{-1} < \rho < \omega_{max}^{-1}$$

Reduce Form

Recall that for ease of interpretation, it is common practice to normalize \mathbf{W} such that the elements of each row sum to unity. Since \mathbf{W} is nonnegative, this ensures that all weights are between 0 and 1, and has the effect that the weighting operation can be interpreted as an averaging of neighboring values.

Theorem (Invertibility of Row-Normalized \mathbf{W} matrix)

If \mathbf{W} is row-normalized, then $(\mathbf{I}_n - \rho\mathbf{W})^{-1}$ exists for all $|\rho| < 1$

Reduce Form: Warning

In spite of its popularity, row-normalized weighting has its drawbacks. Row normalizations alter the internal weighting structure of \mathbf{W} so that comparisons between rows become somewhat problematic.

Example

In view of this limitation, it is natural to consider simple scalar normalizations which multiply \mathbf{W} by a single number, say $a \cdot \mathbf{W}$, which removes any measure-unit effect but preserves relations between all rows of \mathbf{W} .

Reduce Form: Other normalizations

In particular let

$$\begin{aligned} a &= \min \{r, c\} \\ r &= \max_i \sum_j |w_{ij}| \quad \text{maximal row sum of the absolute values} \\ c &= \max_j \sum_i |w_{ij}| \quad \text{maximal column sum of the absolute values.} \end{aligned} \tag{3}$$

- Assuming that the elements of \mathbf{W} are nonnegative, $(\mathbf{I}_n - \rho\mathbf{W})$ will be nonsingular for all $|\rho| < 1/a$.
- This normalization has the advantage of ensuring that the resulting spatial weights, w_{ij} , are all between 0 and 1, and hence can still be interpreted as relative influence intensities.

This is an important result because a model which has a weighting matrix which is not row normalized can always be normalized in such a way that the inverse needed to solve the model will exist in an easily established region.

Expected Value

The expectation is given by:

$$\begin{aligned}\mathbb{E}(\mathbf{y}|\mathbf{X}, \mathbf{W}) &= \mathbb{E} \left[(\mathbf{I}_n - \rho\mathbf{W})^{-1} (\alpha\mathbf{1}_n + \mathbf{X}\boldsymbol{\beta}) + (\mathbf{I}_n - \rho\mathbf{W})^{-1} \boldsymbol{\varepsilon} \mid \mathbf{X}, \mathbf{W} \right] \\ &= (\mathbf{I}_n - \rho\mathbf{W})^{-1} (\alpha\mathbf{1}_n + \mathbf{X}\boldsymbol{\beta}).\end{aligned}\tag{4}$$

To understand this expected value, we need to know the **Leontief Expansion**.

Expected Value

Lemma (Leontief Expansion)

If $|\rho| < 1$, then

$$(\mathbf{I} - \rho\mathbf{W})^{-1} = \sum_{i=0}^{\infty} (\rho\mathbf{W})^i$$

Then, using Lemma 4 (Leontief Expansion), the reduced mode can be written as:

$$\begin{aligned} \mathbf{y} &= (\mathbf{I}_n + \rho\mathbf{W} + \rho^2\mathbf{W}^2 + \dots) (\alpha\mathbf{z}_n + \mathbf{X}\beta) + (\mathbf{I}_n + \rho\mathbf{W} + \rho^2\mathbf{W}^2 + \dots) \boldsymbol{\varepsilon}, \\ &= \alpha\mathbf{z}_n + \rho\mathbf{W}\mathbf{z}_n\alpha + \rho^2\mathbf{W}^2\mathbf{z}_n\alpha + \dots + \mathbf{X}\beta + \rho\mathbf{W}\mathbf{X}\beta + \rho^2\mathbf{W}^2\mathbf{X}\beta + \dots \\ &\quad + \boldsymbol{\varepsilon} + \rho\mathbf{W}\boldsymbol{\varepsilon} + \rho^2\mathbf{W}^2\boldsymbol{\varepsilon}. \end{aligned} \quad (5)$$

Expected Value

Expression (5) can be simplified since the infinite series:

$$\alpha \mathbf{z}_n + \rho \mathbf{W} \mathbf{z}_n \alpha + \rho^2 \mathbf{W}^2 \mathbf{z}_n \alpha + \dots \rightarrow \frac{\mathbf{z}_n \alpha}{(1 - \rho)},$$

since α is a scalar, the parameter $|\rho| < 1$, and \mathbf{W} is row-stochastic. By definition $\mathbf{W} \mathbf{z}_n = \mathbf{z}_n$ and therefore $\mathbf{W} \mathbf{W} \mathbf{z}_n = \mathbf{W} \mathbf{z}_n = \mathbf{z}_n$. Consequently, $\mathbf{W}^l \mathbf{z}_n = \mathbf{z}_n$ for $l \geq 0$ (recall that $\mathbf{W}^0 = \mathbf{I}_n$). This allows to write:

$$\mathbf{y} = \frac{1}{(1 - \rho)} \mathbf{z}_n \alpha + \mathbf{X} \boldsymbol{\beta} + \rho \mathbf{W} \mathbf{X} \boldsymbol{\beta} + \rho^2 \mathbf{W}^2 \mathbf{X} \boldsymbol{\beta} + \dots + \boldsymbol{\varepsilon} + \rho \mathbf{W} \boldsymbol{\varepsilon} + \rho^2 \mathbf{W}^2 \boldsymbol{\varepsilon} + \dots$$

This expression allows defining two effects: a multiplier effect affecting the explanatory variables and a spatial diffusion effect affecting the error terms.

Variance-Covariance Matrix

From reduced-form Equation, we derive the variance-covariance matrix of \mathbf{y} :

$$\begin{aligned}\text{Var}(\mathbf{y} | \mathbf{W}, \mathbf{X}) &= \mathbb{E}(\mathbf{y}\mathbf{y}^\top | \mathbf{W}, \mathbf{X}) \\ &= (\mathbf{I}_n - \rho\mathbf{W})^{-1} \mathbb{E}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^\top | \mathbf{W}, \mathbf{X}) (\mathbf{I}_n - \rho\mathbf{W}^\top)\end{aligned}\tag{6}$$

- This variance-covariance matrix is full, which implies that each location is correlated with every other location in the system (decreasing with the order of proximity).
- Moreover, the elements of the diagonal of $\mathbb{E}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^\top | \mathbf{W}, \mathbf{X})$ are not constant.
 - This implies error heterokedasticity.
 - Since we have not assumed anything about the error variance, we can say that $\mathbb{E}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^\top | \mathbf{W}, \mathbf{X})$ is a full matrix, say $\boldsymbol{\Omega}_\varepsilon$.
 - This covers the possibility of heteroskedasticity, spatial autocorrelation, or both. In absence of either of these complications, the variance matrix simplifies to the usual $\sigma^2\mathbf{I}_n$.

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Spatial Durbin Model (SDM)

The DGP is:

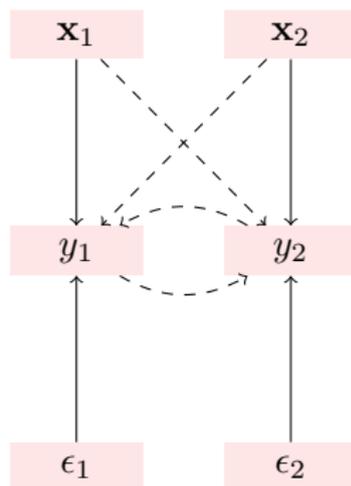
$$\mathbf{y} = \rho \mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \mathbf{W}\mathbf{X}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}$$

$$\mathbf{y} = (\mathbf{I}_n - \rho \mathbf{W})^{-1} (\mathbf{X}\boldsymbol{\beta} + \mathbf{W}\mathbf{X}\boldsymbol{\gamma}) + (\mathbf{I}_n - \rho \mathbf{W})^{-1} \boldsymbol{\varepsilon}$$

The SDM results in a spatial autoregressive model of a special form, including not only the spatially lagged dependent variable and the explanatory variables, but also the spatially lagged explanatory variables, $\mathbf{W}\mathbf{X}$: \mathbf{y} depends on own-regional factors from matrix \mathbf{X} , plus the same factors averaged the n neighboring regions.

Spatial Durbin Model (SDM)

Figure: The SDM for Two Regions



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Spatial Error Model

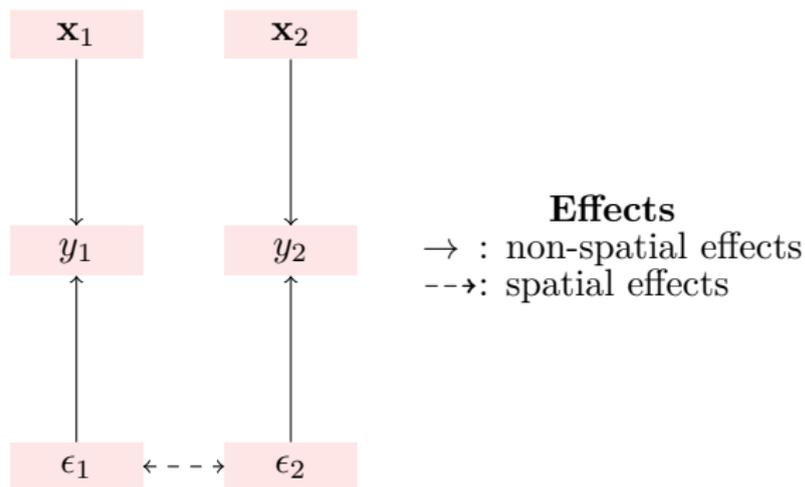
We can also use spatial lags to reflect dependence in the disturbance process, which lead to the spatial error model (SEM):

$$\begin{aligned}\mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \\ \mathbf{u} &= \lambda\mathbf{W}\mathbf{u} + \boldsymbol{\varepsilon} \\ \mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + (\mathbf{I}_n - \lambda\mathbf{W})^{-1}\boldsymbol{\varepsilon}\end{aligned}$$

where λ is the autoregressive parameter for the error lag $\mathbf{W}\mathbf{u}$ (to distinguish the notation from the spatial autoregressive coefficient ρ in a spatial lag model), and $\boldsymbol{\varepsilon}$ is a generally a i.i.d noise.

Spatial Error Model

Figure: The SEM for Two Regions



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Spatial Autocorrelation Model (SAC)

This model contains spatial dependence in both the dependent variable and the disturbances:

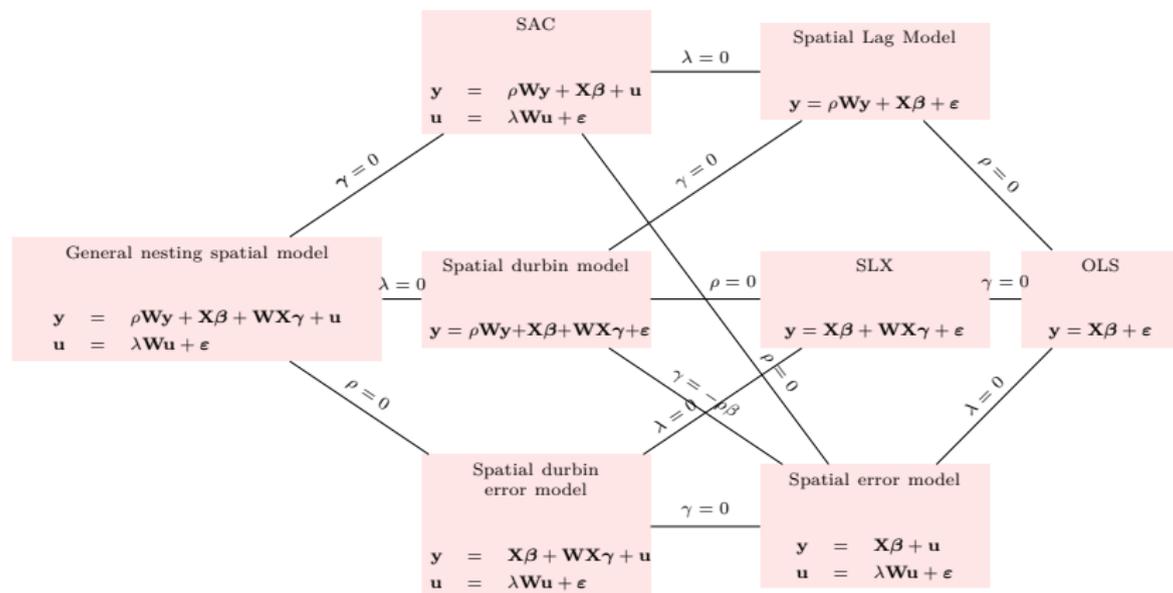
$$\mathbf{y} = \rho \mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

$$\mathbf{u} = \lambda \mathbf{M}\mathbf{u} + \boldsymbol{\varepsilon}$$

$$\mathbf{y} = (\mathbf{I}_n - \rho \mathbf{W})^{-1} \mathbf{X}\boldsymbol{\beta} + (\mathbf{I}_n - \rho \mathbf{W})^{-1} (\mathbf{I}_n - \rho \mathbf{M})^{-1} \boldsymbol{\varepsilon}$$

where \mathbf{W} may be set equal to \mathbf{M}

Elhorts' taxonomy



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Long-run Equilibrium

Consider the following:

- \mathbf{y}_t : dependent variable vector at time t .
- $\mathbf{W}\mathbf{y}_t$: time lag of the average neighboring values of the dependent variable observed during previous period.
- $\mathbf{X}_t = \mathbf{X}$: characteristics of regions remain relatively fixed over time.

Then, the more appropriate process is the following:

$$\mathbf{y}_t = \rho \mathbf{W}\mathbf{y}_{t-1} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_t. \quad (7)$$

Note that we can replace \mathbf{y}_{t-1} on the right-hand side of (7) with:

$$\mathbf{y}_{t-1} = \rho \mathbf{W}\mathbf{y}_{t-2} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_{t-1},$$

producing:

$$y_t = \mathbf{X}\boldsymbol{\beta} + \rho \mathbf{W}(\mathbf{X}\boldsymbol{\beta} + \rho \mathbf{W}\mathbf{y}_{t-2} + \boldsymbol{\varepsilon}_{t-1}) + \boldsymbol{\varepsilon}_t, \quad (8)$$

$$= \mathbf{X}\boldsymbol{\beta} + \rho \mathbf{W}\mathbf{X}\boldsymbol{\beta} + \rho^2 \mathbf{W}^2 \mathbf{y}_{t-2} + \boldsymbol{\varepsilon}_t + \rho \mathbf{W}\boldsymbol{\varepsilon}_{t-1}. \quad (9)$$

Long-run Equilibrium

Recursive substitution for past values of the vector \mathbf{y}_{t-r} on the right-hand side of (9) over q periods leads to:

$$\begin{aligned}\mathbf{y}_t &= (\mathbf{I}_n + \rho\mathbf{W} + \rho^2\mathbf{W}^2 + \dots + \rho^{q-1}\mathbf{W}^{q-1}) \mathbf{X}\boldsymbol{\beta} + \rho^q\mathbf{W}^q\mathbf{y}_{t-q} + \mathbf{u}, \\ \mathbf{u} &= \boldsymbol{\varepsilon}_t + \rho\mathbf{W}\boldsymbol{\varepsilon}_{t-1} + \rho^2\mathbf{W}^2\boldsymbol{\varepsilon}_{t-2} + \dots + \rho^{q-1}\mathbf{W}^{q-1}\boldsymbol{\varepsilon}_{t-(q-1)}.\end{aligned}$$

Noting that:

$$\mathbf{E}(\mathbf{y}_t) = (\mathbf{I}_n + \rho\mathbf{W} + \rho^2\mathbf{W}^2 + \dots + \rho^{q-1}\mathbf{W}^{q-1}) \mathbf{X}\boldsymbol{\beta} + \rho^q\mathbf{W}^q\mathbf{y}_{t-q}, \quad (10)$$

where we use the fact that $\mathbb{E}(\boldsymbol{\varepsilon}_{t-r}) = \mathbf{0}$, $r = 0, \dots, q-1$, which also implies that $\mathbb{E}(\mathbf{u}) = \mathbf{0}$.

Long-run Equilibrium

inally, taking the limit of (10),

$$\lim_{q \rightarrow \infty} \mathbf{E}(\mathbf{y}_t) = (\mathbf{I}_n - \rho \mathbf{W})^{-1} \mathbf{X} \boldsymbol{\beta}. \quad (11)$$

Note that we use the fact that the magnitude of $\rho^q \mathbf{W}^q \mathbf{y}_{t-q}$ tends to zero for large q , under the assumption that $|\rho| < 1$ and assuming that \mathbf{W} is row-stochastic, so the matrix \mathbf{W} has a principal eigenvalue of 1.

Key point:

Equation (11) states that we can interpret the observed cross-sectional relation as the outcome or expectation of a long-run equilibrium or steady state. Note that this provides a dynamic motivation for the data generating process of the cross-sectional SLM that serves as a **workhorse** of spatial regression modeling. That is, a cross-sectional SLM relation can arise from time-dependence of decisions by economic agents located at various point in space when decisions depend on those neighbors.

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SEM and Omitted Variables Motivation

Consider the following process:

$$\mathbf{y} = \mathbf{x}\beta + \mathbf{z}\theta,$$

where \mathbf{x} and \mathbf{z} are **uncorrelated** vectors of dimension $n \times 1$, and the vector \mathbf{z} follows the following spatial autoregressive process:

$$\begin{aligned}\mathbf{z} &= \rho\mathbf{W}\mathbf{z} + \mathbf{r} \\ \mathbf{z} &= (\mathbf{I}_n - \rho\mathbf{W})^{-1} \mathbf{r}\end{aligned}$$

where $\mathbf{r} \sim N(0, \sigma_\varepsilon^2 \mathbf{I}_n)$. Examples of \mathbf{z} are culture, social capital, neighborhood prestige.

If \mathbf{z} is not observed, then:

$$\begin{aligned}\mathbf{y} &= \mathbf{x}\beta + \mathbf{u} \\ \mathbf{u} &= (\mathbf{I}_n - \rho\mathbf{W})^{-1} \boldsymbol{\varepsilon}\end{aligned}\tag{12}$$

where $\boldsymbol{\varepsilon} = \theta\mathbf{r}$. Then, we have the DGP for the spatial error model.

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SDM and Omitted Variables Motivation

Now suppose that \mathbf{X} and $\boldsymbol{\varepsilon}$ from (12) are correlated, given by the following process:

$$\begin{aligned}\boldsymbol{\varepsilon} &= \mathbf{x}\boldsymbol{\gamma} + \mathbf{v} \\ \mathbf{v} &\sim N(0, \sigma^2 \mathbf{I}_n)\end{aligned}\tag{13}$$

where the scalar parameters $\boldsymbol{\gamma}$ and σ^2 govern the strength of the relationship between \mathbf{X} and $\mathbf{z} = (\mathbf{I}_n - \rho \mathbf{W})^{-1} \mathbf{r}$. Interting (13) into (12), we obtain:

$$\begin{aligned}\mathbf{y} &= \mathbf{x}\boldsymbol{\beta} + (\mathbf{I}_n - \rho \mathbf{W})^{-1} \boldsymbol{\varepsilon} \\ &= \mathbf{x}\boldsymbol{\beta} + (\mathbf{I}_n - \rho \mathbf{W})^{-1} (\mathbf{x}\boldsymbol{\gamma} + \mathbf{v}) \\ &= \mathbf{x}\boldsymbol{\beta} + (\mathbf{I}_n - \rho \mathbf{W})^{-1} \mathbf{x}\boldsymbol{\gamma} + (\mathbf{I}_n - \rho \mathbf{W})^{-1} \mathbf{v}\end{aligned}\tag{14}$$
$$\begin{aligned}(\mathbf{I}_n - \rho \mathbf{W}) \mathbf{y} &= (\mathbf{I}_n - \rho \mathbf{W}) \mathbf{x}\boldsymbol{\beta} + \mathbf{v} \\ \mathbf{y} &= \rho \mathbf{W} \mathbf{y} + \mathbf{x}(\boldsymbol{\beta} + \boldsymbol{\gamma}) + \mathbf{W} \mathbf{x}(-\rho \boldsymbol{\beta}) + \mathbf{v}\end{aligned}$$

This is the Spatial Durbin Model (SDM), which includes a spatial lag of the dependent variable \mathbf{y} , as well as the explanatory variables \mathbf{x}

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Global and local spillovers

Spillovers: Key in Regional Science.

Spillovers

A basic definition of spillovers in a spatial context would be that changes occurring in one region exert impacts on other regions

- Changes in tax rate by one spatial unit might exert an impact on tax rate setting decisions of nearby regions, a phenomenon that has been labeled tax mimicking and yardstick competition between local government (see our example below).
- Situations where home improvements made by one homeowner exert a beneficial impact on selling prices of neighboring homes.
- Innovation by university researchers diffuses to nearby firms.
- Air or water pollution generated in one region spills over to nearby regions.

Global and local spillovers

Definition (Global Spillovers)

Global spillovers arise when changes in a characteristic of one region impact all regions' outcomes. This applies even to the region itself since impacts can pass to the neighbors and back to the own region (feedback). Specifically, global spillovers impact the neighbors, neighbors to the neighbors, neighbors to the neighbors to the neighbors and so on.

The endogenous interactions produced by global spillovers lead to a scenario where changes in one region set in motion a sequence of adjustments in (potentially) all regions in the sample such that a new long-run steady state equilibrium arises.

Global and local spillovers

Definition (Local Spillovers)

Local spillovers represent a situation where the impact fall only on nearby or immediate neighbors, dying out before they impact regions that are neighbors to the neighbors.

The main difference is that feedback or endogenous interaction is only possible for global spillovers.

Marginal Effects

- Mathematically, the notion of spillover can be thought as the derivative $\partial y_i / \partial x_j$. This means that changes to explanatory variables in region i impact the dependent variable in region $j \neq i$.
- In the OLS model we have $\partial y_i / \partial x_j = 0$

Measuring spillovers

Consider the SDM model, which can be re-written as:

$$\begin{aligned}(\mathbf{I}_n - \rho \mathbf{W})\mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + \mathbf{W}\mathbf{X}\boldsymbol{\theta} + \boldsymbol{\varepsilon} \\ \mathbf{y} &= \sum_{r=1}^k \mathbf{S}_r(\mathbf{W})\mathbf{x}_r + \mathbf{A}(\mathbf{W})^{-1}\boldsymbol{\varepsilon}\end{aligned}\tag{15}$$

where $\mathbf{S}_r = \mathbf{A}(\mathbf{W})^{-1}(\mathbf{I}_n\beta_r + \mathbf{W}\theta_r)$, and

$$\mathbf{x}_r = \begin{pmatrix} x_{r1} \\ x_{r2} \\ \vdots \\ x_{rn} \end{pmatrix}$$

Measuring spillovers

Now, consider the expansion of the expected value:

$$\begin{pmatrix} \mathbb{E}(y_1) \\ \mathbb{E}(y_2) \\ \vdots \\ \mathbb{E}(y_n) \end{pmatrix} = \sum_{r=1}^k \begin{pmatrix} S_r(\mathbf{W})_{11} & S_r(\mathbf{W})_{12} & \dots & S_r(\mathbf{W})_{1n} \\ S_r(\mathbf{W})_{21} & S_r(\mathbf{W})_{22} & \dots & S_r(\mathbf{W})_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ S_r(\mathbf{W})_{n1} & S_r(\mathbf{W})_{n2} & \dots & S_r(\mathbf{W})_{nn} \end{pmatrix} \begin{pmatrix} x_{1r} \\ x_{2r} \\ \vdots \\ x_{nr} \end{pmatrix} \quad (16)$$

For a single dependent variable, this would be:

$$\mathbb{E}(y_i) = \sum_{r=1}^k [S_r(\mathbf{W})_{i1}x_{1r} + \dots + S_r(\mathbf{W})_{n1}x_{nr}] \quad (17)$$

Measuring spillovers

- 1 Indirect effects: The impact on the expected value of location i given a change in the explanatory variable x_k in location j is

$$\frac{\partial \mathbb{E}(y_i)}{\partial x_{jr}} = S_r(\mathbf{W})_{ij} \quad (18)$$

where $S_r(\mathbf{W})_{ij}$ is this equation represents the i, j th element of the matrix $\mathbf{S}_r(\mathbf{W})$.

- 2 Direct effects: The impact of the expected value of region i , given a change in certain variable for the same region is given by

$$\frac{\partial \mathbb{E}(y_i)}{\partial x_{ir}} = S_r(\mathbf{W})_{ii} \quad (19)$$

This impact includes the **effect of feedback loops** where observation i affects observation j and observation j also affects observation i : a change in x_{ir} will affect the expected value of dependent variable in i , then will pass through the neighbors of i and back to the region itself.

Measuring spillovers

Let us write the all the marginal effects in matrix notation as follows:

$$\begin{aligned} \left(\begin{array}{cccc} \frac{\partial \mathbb{E}(\mathbf{y})}{\partial x_{1r}} & \frac{\partial \mathbb{E}(\mathbf{y})}{\partial x_{2r}} & \cdots & \frac{\partial \mathbb{E}(\mathbf{y})}{\partial x_{nr}} \\ & n \times n & & \end{array} \right) &= \begin{pmatrix} \frac{\partial \mathbb{E}(y_1)}{\partial x_{1r}} & \frac{\partial \mathbb{E}(y_1)}{\partial x_{2r}} & \cdots & \frac{\partial \mathbb{E}(y_1)}{\partial x_{nr}} \\ \frac{\partial \mathbb{E}(y_2)}{\partial x_{1r}} & \frac{\partial \mathbb{E}(y_2)}{\partial x_{2r}} & \cdots & \frac{\partial \mathbb{E}(y_2)}{\partial x_{nr}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbb{E}(y_n)}{\partial x_{1r}} & \frac{\partial \mathbb{E}(y_n)}{\partial x_{2r}} & \cdots & \frac{\partial \mathbb{E}(y_n)}{\partial x_{nr}} \end{pmatrix} \\ &= \mathbf{A}(\mathbf{W})^{-1} (\mathbf{I}_n \beta_r + \mathbf{W} \theta_r) = \mathbf{S}_r(\mathbf{W}) \\ &= (\mathbf{I}_n - \rho \mathbf{W})^{-1} \begin{pmatrix} \beta_r & w_{12} \theta_r & \cdots & w_{1n} \theta_r \\ w_{21} \theta_r & \beta_r & \cdots & w_{2n} \theta_r \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} \theta_r & w_{n2} \theta_r & \cdots & \beta_r \end{pmatrix} \end{aligned}$$

Measuring spillovers

Following Elhorst (2010), consider:

$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 0 \\ w_{21} & 0 & w_{23} \\ 0 & 1 & 0 \end{pmatrix} \quad (20)$$

and

$$\mathbf{A}(\mathbf{W})^{-1} = \frac{1}{1 - \rho^2} \begin{pmatrix} 1 - w_{23}\rho^2 & \rho & \rho^2 w_{23} \\ \rho w_{21} & 1 & \rho w_{23} \\ \rho^2 w_{21} & \rho & 1 - w_{21}\rho^2 \end{pmatrix} \quad (21)$$

Substituting Equations 20 and 21 into Equation 37 we get:

$$\left(\frac{\partial \mathbf{E}(\mathbf{y})}{\partial x_{1r}} \quad \frac{\partial \mathbf{E}(\mathbf{y})}{\partial x_{2r}} \quad \dots \quad \frac{\partial \mathbf{E}(\mathbf{y})}{\partial x_{nr}} \right) = \frac{1}{1 - \rho^2} \begin{pmatrix} (1 - w_{23}\rho^2)\beta_r + (w_{21}\rho)\theta_r & \rho\beta_r + \theta_r & (w_{23}\rho^2)\beta_r + (\rho w_{23})\theta_r \\ (w_{21}\rho)\beta_r + w_{21}\theta_r & \beta_r + \rho\theta_r & (w_{23}\rho)\beta_r + w_{23}\theta_r \\ (w_{21}\rho^2)\beta_r + (w_{21}\rho)\theta_r & \rho\beta_r + \theta_r & (1 - w_{21}\rho^2)\beta_r + (w_{23}\rho)\theta_r \end{pmatrix}$$

Measuring spillovers

- Note that direct effects = diagonal of previous matrix.
- Indirect effects = every non-diagonal elements.
 - What happens with the IEs if $\rho = 0$ and $\theta_r = 0$?
- The direct and indirect effects are different for different units in the sample.
- IEs that occur if $\theta_r \neq 0$ are known as **local effects**, as opposed to IE that occurs if $\rho \neq 0$ and that are known as **global effects**.
 - Local effects: If the element w_{ij} is non-zero (zero), then the effect of x_{jk} on y_i is also non-zero (zero). Note that:

$$\frac{\partial \mathbb{E}(y_3)}{\partial x_{1r}} = \frac{1}{1 - \rho^2} (w_{21}\rho^2\beta_r + w_{21}\rho\theta_r)$$

therefore, the impact of x_{1r} on y_3 will depend on the global effect $(w_{21}\rho^2\beta_r)/(1 - \rho^2)$ and the local effect $w_{21}\rho\theta_r/(1 - \rho^2)$

- What if $w_{21} = 0$? what if ρ increases?

Summary Measures

- It can be noted that the change of each variable in each region implies n^2 potential marginal effects.
- If we have K variables in our model, this implies $K \times n^2$ potential measures.
- Even for small values of n and K , it may already be rather difficult to report these results compactly

We need summary measures!

Summary Measures

Definition (Average Direct Impact)

Let $\mathbf{S}_r = \mathbf{A}(\mathbf{W})^{-1} (\mathbf{I}_n \beta_r + \mathbf{W} \theta_r)$ for variable r . The impact of changes in the i th observation of x_r , which is denoted x_{ir} , on y_i could be summarized by measuring the average $S_r(\mathbf{W})_{ii}$, which equals

$$\text{ADI} = \frac{1}{n} \text{tr}(\mathbf{S}_r(\mathbf{W})) \quad (22)$$

Averaging over the direct impact associated with all observations i is similar in spirit to typical regression coefficient interpretations that represent average response of the dependent to independent variables over the sample of observations.

Summary Measures

Definition (Average Total Impact to an Observation)

Let $\mathbf{S}_r = \mathbf{A}(\mathbf{W})^{-1} (\mathbf{I}_n \beta_r + \mathbf{W} \theta_r)$ for variable r . The sum across the i th row of $\mathbf{S}_r(\mathbf{W})$ would be represent the total impact on individual observation y_i resulting from changing the r th explanatory variable by the same amount across all n observations. There are n of these sums given by the column vector $\mathbf{c}_r = \mathbf{S}_r(\mathbf{W}) \mathbf{1}_n$, so an average of these total impacts is:

$$\text{ATIT} = \frac{1}{n} \mathbf{1}'_n \mathbf{c}_r \quad (23)$$

Definition (Average Total Impact from an Observation)

Let $\mathbf{S}_r = \mathbf{A}(\mathbf{W})^{-1} (\mathbf{I}_n \beta_r + \mathbf{W} \theta_r)$ for variable r . The sum down the j th column of $\mathbf{S}_r(\mathbf{W})$ would yield the total impact over all y_i from changing the r th explanatory variable by an amount in the j th observation. There are n of these sums given by the row vector $\mathbf{r}_r = \mathbf{1}'_n \mathbf{S}_r(\mathbf{W})$, so an average of these total impacts is:

$$\text{ATIF} = \frac{1}{n} \mathbf{r}_r \mathbf{1}_n \quad (24)$$

Reporting Direct and Indirect Effects

We might compute the following measures representing the average total impacts, the average direct impacts, and the average indirect impacts from changes in the model variable X_r

$$\begin{aligned}\bar{M}(r)_{\text{direct}} &= n^{-1} \text{tr}(S_r(W)) \\ \bar{M}(r)_{\text{total}} &= n^{-1} \mathbf{1}' S_r(W) \mathbf{1} \\ \bar{M}(r)_{\text{indirect}} &= \bar{M}(r)_{\text{total}} - \bar{M}(r)_{\text{direct}}\end{aligned}$$

These measures are **inefficient** since we require inversion of the $n \times n$ matrix $(\mathbf{I}_n - \rho \mathbf{W})$.

Reporting Direct and Indirect Effects

Remark These effects reflect how these changes would work through the simultaneous dependence system over time to culminate in a new steady state equilibrium!

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- 4 Empirical Example: Kirby and Lesage, 2009.
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Reporting Direct and Indirect Effects

- These scalar summary measures of impact reflect how these changes would work through the simultaneous dependence system over time to culminate in a new steady state equilibrium.
- Impacts that would take place once all regions reach their equilibrium after the initial change in the variable of interest.
- However one could track the cumulative effects as the impacts pass through neighbors, neighbors of neighbors and so on.

$$\left(\frac{\partial \mathbf{E}(\mathbf{y})}{\partial x_{1r}} \quad \frac{\partial \mathbf{E}(\mathbf{y})}{\partial x_{2r}} \quad \dots \quad \frac{\partial \mathbf{E}(\mathbf{y})}{\partial x_{nr}} \right) \approx (\mathbf{I}_n + \rho \mathbf{W} + \rho^2 \mathbf{W}^2 + \rho^3 \mathbf{W}^3 + \dots + \rho^l \mathbf{W}^l) \mathbf{I}_n \beta_r \quad (25)$$

This expression allow us to observe the impact associated with each power of \mathbf{W} , where these powers corresponds to the observation themselves (zero-order), immediate neighbors (first-order), neighbors of neighbors (second-order), and so on. Using this expansion we could account for both the cumulative effects as marginal and total direct, indirect associated with different order of neighbors.

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Lesage's Example of Spillovers



(a) Professor James Lesage



(b) Professor R Kelley Pace

Figure: Introduction to Spatial Econometrics

Lesage's Example of Spillovers

Consider the following example,

R1	R2	R3	R4 CBD	R5	R6	R7
----	----	----	-----------	----	----	----

West

Highway

East

R1	R2	R3	R4 CBD	R5	R6	R7
----	----	----	-----------	----	----	----

Lesage's Example of Spillovers

Consider the following example,

$$y = \begin{pmatrix} \text{Travel times} \\ 42 \\ 37 \\ 30 \\ 26 \\ 30 \\ 37 \\ 42 \end{pmatrix} \quad X = \begin{pmatrix} \text{Density} & \text{Distance} \\ 10 & 30 \\ 20 & 20 \\ 30 & 10 \\ 50 & 0 \\ 30 & 10 \\ 20 & 20 \\ 10 & 30 \end{pmatrix} \begin{matrix} \text{ex-urban areas } R1 \\ \text{far suburbs } R2 \\ \text{near suburbs } R3 \\ \text{CBD } R4 \\ \text{near suburbs } R5 \\ \text{far suburbs } R6 \\ \text{ex-urban areas } R7 \end{matrix}$$

Lesage's Example of Spillovers

- Consider

$$\mathbf{y} = \rho \mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \quad (26)$$

such that:

$$\hat{\mathbf{y}} = (\mathbf{I}_n - \hat{\rho}\mathbf{W})^{-1}\mathbf{X}\hat{\boldsymbol{\beta}} \quad (27)$$

- Somehow, we get the following estimates:
 - $\hat{\boldsymbol{\beta}} = (0.135 \quad 0.561)'$
 - $\hat{\rho} = 0.642$. It indicates positive spatial dependence in the commuting times.
- What would happen if we double the density of region 2?

Lesage's Example of Spillovers

```
# Estimated coefficients
b <- c(0.135, 0.561)
rho <- 0.642

# W and X
X <- cbind(c(10, 20, 30, 50, 30, 20, 10),
           c(30, 20, 10, 0, 10, 20, 30))
W <- cbind(c(0, 1, 0, 0, 0, 0, 0),
           c(1, 0, 1, 0, 0, 0, 0),
           c(0, 1, 0, 1, 0, 0, 0),
           c(0, 0, 1, 0, 1, 0, 0),
           c(0, 0, 0, 1, 0, 1, 0),
           c(0, 0, 0, 0, 1, 0, 1),
           c(0, 0, 0, 0, 0, 1, 0))
Ws <- W / rowSums(W)

# Prediction
yhat_1 <- solve(diag(nrow(W)) - rho * Ws) %*% crossprod(t(X), b)
```

Lesage's Example of Spillovers

```
# Now we double the population density of a single region
X_d <- cbind(c(10, 40, 30, 50, 30, 20, 10),
            c(30, 20, 10, 0, 10, 20, 30))
yhat_2 <- solve(diag(nrow(W)) - rho * Ws) %*% crossprod(t(X_d), b)

result <- cbind(yhat_1, yhat_2, yhat_2 - yhat_1)
colnames(result) <- c("y1", "y2", "y2 - y1")
round(result, 2)

##           y1      y2 y2 - y1
## [1,] 41.90 44.46    2.56
## [2,] 36.95 40.93    3.99
## [3,] 29.84 31.28    1.45
## [4,] 25.90 26.43    0.53
## [5,] 29.84 30.03    0.19
## [6,] 36.95 37.03    0.08
## [7,] 41.90 41.95    0.05

sum(yhat_2 - yhat_1)

## [1] 8.846915
```

Lesage's Example of Spillovers

```
# Ols prediction
b_ols <- c(0.55, 1.25)
yhat_1 <- crossprod(t(X), b_ols)
yhat_2 <- crossprod(t(X_d), b_ols)
result <- cbind(yhat_1, yhat_2, yhat_2 - yhat_1)
colnames(result) <- c("y1", "y2", "y2 - y1")
round(result, 2)
```

```
##           y1    y2 y2 - y1
## [1,] 43.0 43.0      0
## [2,] 36.0 47.0     11
## [3,] 29.0 29.0      0
## [4,] 27.5 27.5      0
## [5,] 29.0 29.0      0
## [6,] 36.0 36.0      0
## [7,] 43.0 43.0      0
```

Lesage's Example of Spillovers

We know that the impact of changes in the i th observation of x_r on y_i is:

$$\frac{\partial y_i}{\partial x_{ir}} = S_r(W)_{ii} \quad (28)$$

```
S <- solve(diag(nrow(W)) - rho * Ws) %*% diag(nrow(W)) * 0.135
round(diag(S), 4)

## [1] 0.1761 0.1993 0.1792 0.1769 0.1792 0.1993 0.1761
```

Then, the direct impact for R2 is:

$$\Delta CT_2 = S_r(W)_{22} \Delta \text{density}_2 = S_r(W)_{22} \cdot 20 \quad (29)$$

```
S[2,2] * 20

## [1] 3.986773
```

Lesage's Example of Spillovers

We know that:

$$\frac{\partial y_i}{\partial x_{jr}} = S_r(W)_{ij} \quad (30)$$

Then:

$$\Delta CT_1 = S_r(W)_{12} \Delta \text{density}_2 = S_r(W)_{12} \cdot 20 \quad (31)$$

```
S[1,2] * 20
```

```
## [1] 2.559508
```

Lesage's Example of Spillovers

- **Total impact to an observation:** The sum across the i th row fo $S_r(W)$ would represent the total impact on individual observation y_i resulting from changing the r th explanatory variable by the same among across all n observations $x_r + \delta \mathbf{z}_n$

What would be the impact on commuting time on R1 if density increases by 20 in all the Regions?

```
sum(S[1, ]) * 20
```

```
## [1] 7.541899
```

This number implies that the total impact to R1 will be an increase of commuting time of ≈ 7.5 minutes.

Lesage's Example of Spillovers

- **Total impact from an observation:** The sum down the j th column of $S_r(W)$ would yield the total impact over all y_i from changing the r th explanatory variable by an amount in the j th observation (e.g., $x_{jr} + \delta$)
- What would be the impact of increasing density by 20 in R2 on all the regions?

```
sum(S[, 2]) * 20  
## [1] 8.846915
```

- Note that this is the same number that we get from the previous table.
- From which region will the total impact higher?

```
round(colSums(S), 2)  
## [1] 0.28 0.44 0.40 0.39 0.40 0.44 0.28
```

Note that an increase in 1 in density will have a greater impact on all the regions if this increase comes from R2 and R6 (Why?).

Lesage's Example of Spillovers

```
# Average Direct Impact (of Density)
```

```
ADI <- sum(diag(S)) / nrow(W)
```

```
ADI
```

```
## [1] 0.1837338
```

```
# Total
```

```
Total <- crossprod(rep(1, nrow(W)), S) %*% rep(1, nrow(W)) / nrow(W)
```

```
Total
```

```
##           [,1]
```

```
## [1,] 0.377095
```

```
# Indirect
```

```
Total - ADI
```

```
##           [,1]
```

```
## [1,] 0.1933612
```

Cumulative Effects

The main idea of this exercise is to show how the change in some explanatory variable produces changes in the independent variable in all the spatial units by decomposing them into cumulative and marginal impacts for different order of neighbors

Cumulative Effects

```
library("expm") # Package to compute power of a matrix
## Loop for decomposition
n <- nrow(W)
b_dens <- 0.135
out <- matrix(NA, nrow = 11, ncol = 3) # Matrix for the results
colnames(out) <- c("Total", "Direct", "Indirect") # colnames
rownames(out) <- paste("q", sep = "=", seq(0, 10)) # rownames

for (q in 0:10) {
  if (q == 0) { # If q=0, then Sr = I * beta
    S <- diag(n) * b_dens
  } else {
    S <- (rho ^ q * Ws %^% q) * b_dens
  }
  q <- q + 1 # the row = 0 doesn't exist!
  out[q, 2] <- sum(diag(S)) / n
  out[q, 1] <- crossprod(rep(1, n), S) %*% rep(1, n) / n
  out[q, 3] <- out[q, 1] - out[q, 2]
}
```

Cumulative Effects

```
# Print results
```

```
round(out, 4)
```

```
##      Total Direct Indirect
## q=0  0.1350 0.1350  0.0000
## q=1  0.0867 0.0000  0.0867
## q=2  0.0556 0.0318  0.0238
## q=3  0.0357 0.0000  0.0357
## q=4  0.0229 0.0106  0.0123
## q=5  0.0147 0.0000  0.0147
## q=6  0.0095 0.0039  0.0056
## q=7  0.0061 0.0000  0.0061
## q=8  0.0039 0.0015  0.0024
## q=9  0.0025 0.0000  0.0025
## q=10 0.0016 0.0006  0.0010
```

```
round(colSums(out), 4)
```

```
##      Total   Direct Indirect
##      0.3742  0.1834  0.1909
```

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Changes in commuting to work times over the 1990 to 2000 period

- The 1990 to 2000 period shows an increase of workers commuting more than 45 or more minutes.

Debate: do cities act as a dynamic system where flexible land markets work to offset congestion costs that would in the face of urban growth lead to congested and unlivable cities. ?

Contribution

Incorporate congestion spillovers impacts arising from shared roadways using spatial econometrics.

Methodology aspects

Intuition:

Since roadways used to commute via automobile to work represent a shared resource of residents from neighboring tracts, and increase in the number of commuters in one tract will produce a spillover externality on commuters from neighboring tracts.

Spatial Durbin Model

There are some important variables that exert an influence on longer commuting times that have been excludedn from the model. (Joint location decision that must be made by households where both spouses work)

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Data

- Spatial Unit: Census tract in the contiguous US states:
 - 64,111 tracts for 2000 and 63,932 for 1990, after cleaning for zero variables.

The focus is on understanding which population characteristics exerted an important influence on travel to work times.

- Location decisions of the population and resident population characteristics.
 - These variables shed light on the role of housing and land markets that are thought to respond dynamically to congestion costs.
 - Recent location decisions during the last 5 years would seem to be particularly important for explaining longer commute times reported for tract-level population of 45 min or more.
- Focus only on the number of workers over 16 in each Census tract with travel times over 45 min (dependent variables.)
 - This represent only 12.5% of workers over 16 years in 1990 (excluding those who worked at home)
 - 15.4% of workers in 2000.
- Dependent Variable: Log of number of persons in each tract with a travel to work time over 45 min.

Table 3

Location related variables used in the model.

Labels used in tables	Definitions
Work at home	Workers 16+ that work at home
Work in city	Workers 16+ in the metro area, work in the city
Work in suburbs	Workers 16+ in the metro area, work in suburbs
Work in ocity	Workers 16+ in the metro area, work in the city, another metro area
Work in osuburbs	Workers 16+ in the metro area, work in suburbs, another metro area
Work in non-metro	Workers 16+ that work in non-metro areas
Moved from city	Persons 5+ moved from central city, same metro area last 5 years
Moved from ocity	Persons 5+ moved from central city, another metro area last 5 years
Moved from suburbs	Persons 5+ moved from suburbs, the same metro area last 5 years
Moved from nmetro	Persons 5+ moved from non-metro area last 5 years
Moved in 1 year	Housing units where the householder moved in during the last year
Moved in 2-5 years	Housing units where hhdr. moved in during the last 2-5 years
Moved in 6-10 years	Housing units where hhdr. moved in during the last 6-10 years
Moved in 11-19 years	Housing units where hhdr. moved in during the last 11-19 years
Moved in 20-29 years	Housing units where hhdr. moved in during the last 20-29 years
Moved in 30+ years	Housing units where hhdr. moved in over 30 years ago

Results

Let's take a look to the direct effects.

Results from SDM: Direct Effects

Table 6

Sorted direct effects estimates for 1990.

Variable	Direct effect 1990	Standard Deviation	t-stat	t-prob
Renters	-0.0727	0.0066	-11.0776	0.0000
College	-0.0605	0.0050	-12.0589	0.0000
Income	-0.0601	0.0081	-7.3789	0.0000
Walking	-0.0367	0.0034	-10.8444	0.0000
Moved in 30+ years	-0.0253	0.0029	-8.8647	0.0000
Moved from city	-0.0253	0.0043	-5.8750	0.0000
Moved from suburbs	-0.0239	0.0039	-6.1265	0.0000
Moved from ocity	-0.0211	0.0034	-6.1384	0.0000
Moved from nmetro	-0.0112	0.0027	-4.1102	0.0000
Moved in 11-19 years	0.0134	0.0059	2.2828	0.0224
Females 55-64	0.0141	0.0068	2.0847	0.0371
No car	0.0245	0.0039	6.3330	0.0000
Work in non-metro	0.0247	0.0021	11.5062	0.0000
Moved in 1 year	0.0299	0.0082	3.6657	0.0002
Males 25-29	0.0328	0.0080	4.0961	0.0000
Water	0.0333	0.0046	7.2438	0.0000
Work in suburbs	0.0337	0.0043	7.9028	0.0000
Females 30-34	0.0442	0.0096	4.6188	0.0000
Land	0.0486	0.0024	20.0071	0.0000
Work in city	0.0518	0.0053	9.7863	0.0000
Work in ocity	0.0519	0.0021	24.2134	0.0000
Males 35-44	0.0519	0.0116	4.4827	0.0000
Public trans	0.0563	0.0020	27.8689	0.0000
Females 35-44	0.0569	0.0127	4.4672	0.0000
Work in osuburbs	0.0590	0.0020	28.8100	0.0000
Females 25-29	0.0658	0.0088	7.5098	0.0000
Males 45-54	0.0739	0.0084	8.7706	0.0000
Moved in 2-5 years	0.0814	0.0101	8.0250	0.0000
Pop	0.0980	0.0203	4.8356	0.0000
Employed females	0.1306	0.0275	4.7510	0.0000
Employed males	0.2742	0.0328	8.3598	0.0000

Results from SDM

- Model in the log-log form.
 - The direct effects estimates reflect the percentage increase in the number of workers (within a tract, on average) with travel to work times greater than 45 min to a percentage change in the number of residents of the tract having the various explanatory variable characteristics.
 - For example, the direct effect magnitude of -0.0727 for Renters indicates that a 1% increase in the number of workers residing in the tract would lead to a decrease of 0.07% in the number of workers residing in the tract that have travel times to work greater than 45.
- Changes in the tract-level residents representing Renters exerts the greatest impact on reducing (own-tract) residents having longer commuting times, followed by tract residents with college or post college educational attainment and the median household income of tract residents.
- It seems plausible that renters might be better positioned to locate optimally with respect to work locations to avoid longer commute times.
- Both population and land area have positive and significant effects, with the population effect greater than that for land area. Population density: $\log(\text{pop}) - \log(\text{land})$.

Results from SDM: Indirect Effects

- Negative effects could be considered spatial benefits, since these indicate variables that lead to a reduction in persons in neighboring tracts with longer commuting times.
- Positive indirect effects would represent a negative externality, since this indicates that neighboring tracts suffer from an increase in residents with longer commute times when these variables increase.

Results from SDM: Indirect Effects

Table 7

Sorted indirect effects estimates for 1990.

Variable	Indirect effect 1990	Standard Deviation	t-stat	t-prob
Employed males	-0.9802	0.1382	-7.0919	0.0000
Males 55-64	-0.1925	0.0522	-3.6855	0.0002
Moved from ocity	-0.1721	0.0147	-11.6713	0.0000
Moved in 30+ years	-0.1628	0.0122	-13.3154	0.0000
Renters	-0.1519	0.0274	-5.5393	0.0000
Walking	-0.1372	0.0151	-9.0810	0.0000
Moved from city	-0.1334	0.0161	-8.2606	0.0000
Females 20-24	-0.1272	0.0440	-2.8902	0.0039
Work in non-metro	-0.0980	0.0082	-11.9873	0.0000
Moved from nmetro	-0.0890	0.0123	-7.2418	0.0000
Three cars	-0.0485	0.0180	-2.6885	0.0072
Work in suburbs	-0.0409	0.0157	-2.6035	0.0092
Water	0.0354	0.0151	2.3453	0.0190
Work in osuburbs	0.0511	0.0071	7.2377	0.0000
Work in city	0.0575	0.0160	3.5875	0.0003
Work at home	0.0658	0.0129	5.1038	0.0000
Land	0.0927	0.0086	10.8247	0.0000
Public trans	0.1344	0.0077	17.4669	0.0000
Females 25-29	0.1805	0.0495	3.6454	0.0003
No car	0.1970	0.0178	11.0396	0.0000
Income	0.3097	0.0403	7.6851	0.0000
Males 45-54	0.3406	0.0457	7.4530	0.0000
Pop	0.4262	0.0936	4.5519	0.0000
Employed females	0.6305	0.1197	5.2679	0.0000

Results from SDM: Indirect Effects

- Renters -0.1519: The negative spillover is larger than the negative direct effects, so renters provides a beneficial reduction in persons with longer commute times for both own- and other-tract resident.
- This seems consistent with the notion that residents who rent rather than own homes may be more capable of choosing locations that reduce travel times to work, and this provides positive spillovers.
- Those who moved within the last 5 years from another city outside the metropolitan area.
 - Recent moves might reflect decisions made to minimize longer travel times to work, and these would likely lead to beneficial spillover effects.

A comparison of 1990 and 2000: Direct Effects

Table 9

Significant changes in direct effects estimates over the 1990 to 2000 period.

Variable	Direct effect 1990	Direct effect 2000	Change 2000-1990	t-stat	t-prob
<i>Part 1: Significant negative changes in direct effects</i>					
Males 45-54	0.0739	-0.0067	-0.0806	-8.5745	0.0000 *
Females 25-29	0.0658	0.0356	-0.0302	-3.8228	0.0001
Males 25-29	0.0328	0.0117	-0.0211	-2.9306	0.0034
Work in suburbs	0.0337	0.0201	-0.0136	-3.2381	0.0012
No car	0.0245	0.0109	-0.0136	-3.5789	0.0003
College	-0.0605	-0.0722	-0.0117	-2.3636	0.0181
Work in non-metro	0.0247	0.0148	-0.0099	-5.0769	0.0000
Work in osuburbs	0.0590	0.0496	-0.0094	-4.9474	0.0000
<i>Part 2: Significant positive changes in direct effects</i>					
Moved from nmetro	-0.0112	-0.0013	0.0099	4.0408	0.0001
Moved in 20-30 years	-0.0071	0.0062	0.0133	3.5467	0.0004 *
Walking	-0.0367	-0.0228	0.0139	4.6333	0.0000
Renters	-0.0727	-0.0533	0.0194	3.2605	0.0011
Females 35-44	0.0569	0.0931	0.0362	2.9431	0.0033
Employed males	0.2742	0.3764	0.1022	4.3489	0.0000

*Indicates a change in sign of the effect between 1990 and 2000.

A comparison of 1990 and 2000: Indirect Effects

Table 10

Significant changes in indirect effects estimates over the 1990 to 2000 period.

Variable	Indirect effect 1990	Indirect effect 2000	Change 2000-1990	t-stat	t-prob
<i>Part 1: Significant negative changes in indirect effects</i>					
Employed Females	0.6305	-0.2845	-0.9150	-10.1329	0.0000*
Males 45-54	0.3406	-0.1205	-0.4611	-8.6673	0.0000*
Moved in 1 year	0.0276	-0.1782	-0.2058	-5.2905	0.0000*
Females 25-29	0.1805	0.0007	-0.1798	-4.0134	0.0001
Females 15-19	0.0662	-0.0662	-0.1324	-3.4842	0.0005*
Moved in 6-10 years	-0.0226	-0.1490	-0.1264	-3.5258	0.0004
Income	0.3097	0.1979	-0.1118	-2.9382	0.0033
No car	0.1970	0.1097	-0.0873	-4.7836	0.0000
<i>Part 2: Significant positive changes in indirect effects</i>					
Moved in 20-30 years	-0.0301	0.0235	0.0536	2.9370	0.0033*
Walking	-0.1372	-0.0823	0.0549	3.9075	0.0001
Moved from metro	-0.0890	-0.0232	0.0658	5.9818	0.0000
Females 20-24	-0.1272	0.1059	0.2331	6.1101	0.0000*
Moved in 2-5 years	0.0737	0.3457	0.2720	5.3333	0.0000
Males 55-64	-0.1925	0.1217	0.3142	6.2465	0.0000*
Females 35-44	-0.0413	0.4170	0.4583	6.5378	0.0000*
Employed males	-0.9802	-0.1551	0.8251	8.1131	0.0000

*Indicates a change in sign of the effect between 1990 and 2000.