Universidad Católica del Norte, Economic Department

Econometrics: Problem Set 3

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This homework is due on 26th, April at 18:00 hrs. Please, write your answers clearly.

1 Theory

1. (Uniform Distribution) Let $X_1, X_2, ..., X_n$ be a sequence of random variable distributed iid $U(0, \theta)$, with θ unknown. Let $\widehat{\theta}_n = 2\bar{X}$, where $\bar{X} = (1/n)\sum_{i=1}^n X_i$. (Hint: If $x \sim U(a,b)$, then $\mathbb{E}(x) = (1/2)(a+b)$ and $\text{Var}(x) = (1/12)(b-a)^2$)
   a) Is $\widehat{\theta}_n$ unbiased?
   b) Find $\text{Var}(\widehat{\theta}_n)$.
   c) Show that $\widehat{\theta}_n - \theta = o_p(1)$.
2. (Consistency under other assumptions) Show the following. Suppose:
   a) $y_i = \boldsymbol{x}_i'\boldsymbol{\beta}_0 + \epsilon_i, \quad i = 1, 2, ..., n; \boldsymbol{\beta}_0 \in \mathbb{R}^K$;
   b) $\boldsymbol{X}'\boldsymbol{\varepsilon}/n \xrightarrow{a.s.} \mathbf{0}$;
   c) $\boldsymbol{X}'\boldsymbol{X}/n \xrightarrow{a.s.} \boldsymbol{M}$; finite and positive definite.

   Then $\widehat{\boldsymbol{\beta}}_n$ exists for all n sufficiently large $a.s$, and $\widehat{\boldsymbol{\beta}}_n \xrightarrow{a.s.} \boldsymbol{\beta}_0$.
3. (Distribution of another estimator) Consider the following model with a iid sample:

$$
\begin{aligned}
y_i &= x_i\beta_0 + \epsilon_i, \quad x_i \in \mathbb{R}, \beta_0 \in \mathbb{R} \\
\mathbb{E}(\epsilon_i|x_i) &= 0 \\
\Omega_0 &= \mathbb{E}(x_i^2\epsilon_i^2)
\end{aligned}
$$

Let $\widehat{\beta}$ be the OLS estimate of β_0 with residuals $\widehat{\epsilon}$. Now, consider the following two estimates of Ω_0:

$$\widetilde{\Omega} = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \epsilon_i^2$$
$$\widehat{\Omega} = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \widehat{\epsilon}_i^2$$

a) Find the asymptotic distribution of $\sqrt{n}(\widetilde{\Omega} - \Omega_0)$. What assumptions do you need about the moments of x_i and ϵ_i ?

b) Find the asymptotic distribution of $\sqrt{n}(\widehat{\Omega} - \Omega_0)$.

c) How do you use the assumption $E(\epsilon_i|x_i) = 0$ in your answer in (b)?

d) (WLS) Consider the model $y_i = \boldsymbol{x}_i'\boldsymbol{\beta}_0 + \epsilon_i$, with $\mathbb{E}(\boldsymbol{x}_i \cdot \epsilon_i) = \mathbf{0}$. For a positive function $w(\boldsymbol{x})$, let $w_i = w(\boldsymbol{x}_i)$. Consider the estimator:

$$\widetilde{\boldsymbol{\beta}} = \left(\sum_{i=1}^{n} w_i \boldsymbol{x}_i \boldsymbol{x}_i'\right)^{-1} \left(\sum_{i=1}^{n} w_i \boldsymbol{x}_i y_i\right)$$

Is $\widetilde{\boldsymbol{\beta}}$ consistent for $\boldsymbol{\beta}_0$. If not, under what assumption is $\widetilde{\boldsymbol{\beta}}$ consistent for $\boldsymbol{\beta}_0$? List all the additional assumptions you had to make.

4. (Constraint OLS and Large Sample Properties) Consider the model $y_i = x_{1i}\beta_1 + x_{2i}\beta_2 + \epsilon_i$ such that $\mathbb{E}(\boldsymbol{x}_i \cdot \epsilon_i) = \mathbf{0}$, with n observations. Consider the restriction

$$\frac{\beta_1}{\beta_2} = 2 \qquad (1.1)$$

a) Please find a explicit expression for the constrained least squares estimator $\widetilde{\boldsymbol{\beta}} = (\widetilde{\beta}_1, \widetilde{\beta}_2)$ of $\boldsymbol{\beta}_0$ under the restriction 1.1. (Hint: find a way of inserting 1.1 into the true model and find the estimator using the usual OLS formula for the estimators).

b) Derive the asymptotic distribution of $\widetilde{\beta}_1$ under the assumption that 1.1 is true. List all the additional assumptions you had to make.