Universidad Católica del Norte, Economic Department

Spatial Econometrics: Problem Set 3

Professor: Mauricio Sarrias

October, 2017

This problem set is due on Thursday 26th, October at 18:00. The solutions must be submitted in a **printed form**. **Please include all your codes in the solution.**

1 Theory

1. (Maximum Likelihood Estimator) Let $x_1, ..., x_n$ be an iid sample distributed as $\text{N}(\mu, \sigma^2)$,
   a) Find the log-likelihood function.
   b) Find the score function.
   c) Find the MLE of μ and σ^2.
   d) Show that $\mathbb{E}(\log L_{\boldsymbol{\theta}}) = \mathbf{0}$, where $\boldsymbol{\theta} = (\mu, \sigma^2)$.
   e) Find the Hessian.
   f) Find the Hessian evaluated at $\widehat{\boldsymbol{\theta}}$.
   g) How can we compute the variance of $\widehat{\boldsymbol{\theta}}$?
2. (SEM model estimated by OLS) Consider the following SEM model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \tag{1.1}$$
$$\mathbf{u} = \lambda\mathbf{W}\mathbf{u} + \boldsymbol{\varepsilon} \tag{1.2}$$
$$\boldsymbol{\varepsilon} \sim \text{N}(\mathbf{0}, \sigma^2\mathbf{I}) \tag{1.3}$$

   Show that the OLS estimates $\widehat{\boldsymbol{\beta}}$ is unbiased, but inefficient.

3. (Maximum Likelihood Estimation of SAC Model) Consider the following SAC model with heteroskedastic errors:

$$\begin{aligned}
\mathbf{y} &= \rho \mathbf{W}_1 \mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \mathbf{u} && (1.4)\\
\mathbf{u} &= \lambda \mathbf{W}_2 \mathbf{u} + \boldsymbol{\varepsilon} && (1.5)\\
\boldsymbol{\varepsilon} &\sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Omega}) && (1.6)
\end{aligned}$$

The matrix $\boldsymbol{\Omega}$ is the variance-covariance matrix of the error terms, which is assummed to be known a priori. For example, we can assume that:

$$\mathrm{Var}(\epsilon_i) = \sigma_i^2 = \mathbf{z}_i^\top \boldsymbol{\alpha} \tag{1.7}$$

or

$$\mathrm{Var}(\epsilon_i) = \sigma_i^2 = \exp(\mathbf{z}_i^\top \boldsymbol{\alpha}) \tag{1.8}$$

or more general,

$$\mathrm{Var}(\epsilon_i) = \sigma_i^2 = \mathbf{h}(\mathbf{z}_i^\top \boldsymbol{\alpha}) \tag{1.9}$$

where $\mathbf{h}(\cdot)$ is any function, $\mathbf{z}_i$ is a vector of covariates for each spatial unit, and $\boldsymbol{\alpha}$ is a vector of parameters with element $\alpha_p, p = 0, 1, ..., P$. Therefore, the diagonal elements of the error covariance matrix $\boldsymbol{\Omega}$ are:

$$\boldsymbol{\Omega}_{ii} = \sigma_i^2 = \mathbf{h}_i(\mathbf{z}_i^\top \boldsymbol{\alpha}), \quad \mathbf{h}_i > 0 \tag{1.10}$$

Note that the model has $2 + K + P$ unknown parameters:

$$\boldsymbol{\theta} = (\rho, \boldsymbol{\beta}^\top, \lambda, \boldsymbol{\alpha}^\top)^\top. \tag{1.11}$$

a) Find the Log-likelihood function. Show every step.

b) Find the first order conditions. Show all your work.

4. (Extended SLM Model) Consider the model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \rho_1 \mathbf{W}_1 \mathbf{y} + \rho_2 \mathbf{W}_2 \mathbf{y} + \mathbf{u}, \tag{1.12}$$

where $\mathbf{u}$ has mean and VC matrix of $\mathbf{0}$ and $\sigma^2 \mathbf{I}_n$, respectively, and $\mathbf{W}_1$ and $\mathbf{W}_2$, are observed exogenous weighting matrices.

a) Obtain the likelihood function, and then determine the first order conditions for $\boldsymbol{\beta}$.

b) Assume that $\mathbf{W}_1$ and $\mathbf{W}_2$ are row-normalized. Give a condition which is sufficient for the model to be solved for $\mathbf{y}$ in terms of $\mathbf{X}$ and $\boldsymbol{\varepsilon}$.

2 Applications in R

1. (ML programming) Now you have to program the ML estimation procedure for Question 1 in R including the gradient and the hessian:
    a) Create a function that returns the log-likelihood, the gradient and the Hessian.
    b) Set the seed at `set.seed(123)`, and generate a random variable x such that $x \sim N(1, 2^2)$ for a sample of 100,000 individuals.
    c) Show that your code for the ML can return the estimates very precisely. Use `maxLik` package (and function) for the estimation procedure.
    d) Show that the standard errors of the estimates can be computed using your formula stated in question (1g).
    e) Do the estimates and standard errors vary if we use different algorithms such as Newton-Raphson, BHHH, or BFGS? Show this using your code.

2. (Monte Carlo) Consider the following DGP:

$$
\begin{aligned}
y_i &= \alpha + \beta x_i + u_i \\
u_i &= \lambda \sum_{j=1}^{n} w_{ij} u_j + \epsilon_i \\
\epsilon_i &\sim \mathrm{N}(0, 1)
\end{aligned}
\tag{2.1}
$$

where $\lambda = 0.8$, $\alpha = 0.5$, $\beta = 1$ and $x_i \sim \mathrm{N}(0, 2^2)$. Using a Monte Carlo experiment, show that the $\widehat{\beta}_{OLS}$ is unbiased, but inefficient. For experiment crreate 100 datasets with 225 spatial units. Set the seed at 123.