Universidad Católica del Norte, Economic Department

Spatial Econometrics: Exam 1

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November, 2017

- If you believe a question is unclear, please state how you interpret the question.
- Please, use formal mathematical language wherever possible.
- You must show all work for partial credit to be awarded.
- Total points: 40.

1 Questions

1. Consider the following spatial model:

$$\begin{aligned}\mathbf{y} &= \rho_1 \mathbf{W}_1 \mathbf{y} + \rho_2 \mathbf{W}_2 \mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \\ \mathbf{u} &= \lambda \mathbf{M}\mathbf{u} + \boldsymbol{\varepsilon}\end{aligned}$$

where $\mathbf{W}_1 \neq \mathbf{W}_2 \neq \mathbf{M}$; and $\boldsymbol{\varepsilon} \sim \mathrm{N}(\mathbf{0}, \sigma_\epsilon^2 \mathbf{I})$.

   a) (10 pts) Obtain the log-likelihood function.
   b) (10 pts) Determine $\widehat{\boldsymbol{\beta}}_{ML}$.
   c) (10 pts) Set $\lambda = 0$. Is it the OLS estimate of $\widehat{\boldsymbol{\beta}}$ unbiased?.
   d) (10 pts) Set $\rho_1 = \rho_2 = 0$. How do you estimate λ and $\boldsymbol{\beta}$ by ML?

Some important results are the followings:

$$\frac{\partial(\rho \mathbf{W})}{\partial \rho} = \mathbf{W} \tag{1}$$

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial \rho} &= \frac{\partial(\mathbf{I} - \rho\mathbf{W})}{\partial \rho} \\ &= \frac{\partial \mathbf{I}}{\partial \rho} - \frac{\partial \rho \mathbf{W}}{\rho} \\ &= -\mathbf{W}\end{aligned} \tag{2}$$

$$\frac{\partial \log|\mathbf{A}|}{\partial \rho} = \operatorname{tr}(\mathbf{A}^{-1}\partial\mathbf{A}/\partial\rho) = \operatorname{tr}\left[\mathbf{A}^{-1}(-\mathbf{W})\right] \tag{3}$$

Let $\boldsymbol{\varepsilon} = \mathbf{A}\mathbf{y} - \mathbf{X}\boldsymbol{\beta}$, then:

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial \rho} = \frac{\partial(\mathbf{A}\mathbf{y} - \mathbf{X}\boldsymbol{\beta})}{\partial \rho} = -\mathbf{W}\mathbf{y} \tag{4}$$

$$\frac{\partial \boldsymbol{\varepsilon}^\top \boldsymbol{\varepsilon}}{\partial \rho} = \boldsymbol{\varepsilon}^\top(\partial\boldsymbol{\varepsilon}/\partial\rho) + (\partial\boldsymbol{\varepsilon}^\top/\partial\rho)\boldsymbol{\varepsilon} = 2\boldsymbol{\varepsilon}^\top(\partial\boldsymbol{\varepsilon}/\partial\rho) = 2\boldsymbol{\varepsilon}^\top(-\mathbf{W})\mathbf{y} \tag{5}$$

$$\frac{\partial \mathbf{A}^{-1}}{\partial \rho} = -\mathbf{A}^{-1}(\partial\mathbf{A}/\partial\rho)\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{W}\mathbf{A}^{-1} \tag{6}$$

$$\frac{\partial \operatorname{tr}\left(\mathbf{A}^{-1}\mathbf{W}\right)}{\partial \rho} = \operatorname{tr}\left(\partial\mathbf{A}^{-1}\mathbf{W}/\partial\rho\right) \tag{7}$$